Sweeping a Plane

A developable surface is defined as the envelope surface of a moving plane or, equivalently, that of one-parameter family of planes $\{L(t)\}$.

Let (x, y, z) be a point common to two nearby planes:

$$\begin{aligned} L(t): & \quad a(t)x + b(t)y + c(t)z = d(t) \\ L(t+dt): & \quad a(t+dt)x + b(t+dt)y + c(t+dt)z = d(t+dt) \end{aligned}$$

The point (x, y, z) on their intersection line satisfies the following linear constraint

$$(a(t+dt) - a(t))x + (b(t+dt) - b(t))y + (c(t+dt) - c(t))z = d(t+dt) - d(t),$$

or, for an infinitesimally small value of $dt \neq 0$,

$$\frac{a(t+dt) - a(t)}{dt}x + \frac{b(t+dt) - b(t)}{dt}y + \frac{c(t+dt) - c(t)}{dt}z = \frac{d(t+dt) - d(t)}{dt}.$$

As $dt \to \infty$, we have

$$L_t(t): a'(t)x + b'(t)y + c'(t)z = d'(t).$$

The developable surface defined by the family of planes $\{L(t)\}$ is also a ruled surface, which consists of lines defined by

$$\begin{aligned} L(t): & \quad a(t)x + b(t)y + c(t)z = d(t) \\ L_t(t): & \quad a'(t)x + b'(t)y + c'(t)z = d'(t). \end{aligned}$$

The direction of this ruling line is given by the cross product of the normal vectors of these two planes:

$$\mathbf{n}(t) = (a(t), b(t), c(t)) \times (a'(t), b'(t), c'(t)).$$

Now we consider the construction for a directrix curve on this ruled surface. Assume that $\mathbf{n}(t)$ is not pararallel to the xy-plane; then the intersection of three planes: $L(t)$, $L_t(t)$, and the xy-plane, will generate such a curve:

$$\begin{bmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c'(t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d(t) \\ d'(t) \\ 0 \end{bmatrix}$$

The indicatrix curve $\alpha(t)$ is generated by solving this matrix equation using Cramer's rule. The developable surface itself is given as follows:

$$D(s, t): \ \alpha(t) + s\mathbf{n}(t).$$

Two Parameter Sweeping of a Plane

When a plane is under a two-parameter spline motion, the sweep envelope surface can be constructed as a spline surface. Given a two-parameter family of planes $\{L(s,t)\}$, each point on the envelope surface satisfies the following three linear equationa:

$$\begin{aligned} L(s,t): &\quad a(s,t)x + b(s,t)y + c(s,t)z = d(s,t), \\ L_s(s,t): &\quad a_s(s,t)x + b_s(s,t)y + c_s(s,t)z = d_s(s,t), \\ L_t(s,t): &\quad a_t(s,t)x + b_t(s,t)y + c_t(s,t)z = d_t(s,t). \end{aligned}$$

Equivalently, we have

$$\begin{bmatrix} a(s,t) & b(s,t) & c(s,t) \\ a_s(s,t) & b_s(s,t) & c_s(s,t) \\ a_t(s,t) & b_t(s,t) & c_t(s,t) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d(s,t) \\ d_s(s,t) \\ d_t(s,t) \end{bmatrix}$$

The sweep envelope surface itself can be constructed by solving this matrix equation by Cramer's rule.