Computing Rational Bisectors



Gershon Elber Technion, Israel Institute of Technology

Myung-Soo Kim Seoul National University

G iven two objects, we define their bisector as the set of points equidistant from the two objects. Bisector construction plays an important role in many geometric computations, such as Voronoi diagrams construction, medial axis transformation, shape decomposition, mesh generation, collision-avoidance motion planning, and NC tool path generation,¹ to mention a few.

Bisector construction plays an important role in many geometric computations. This article explains how to compute rational bisectors of point-surface and spheresurface pairs. Unfortunately, the bisector of even simple geometric primitives is not always simple. While the bisector of two lines in the plane is a line, the bisector of two skewed lines in R^3 (3D space) is a hyperbolic paraboloid of one sheet.² The bisector of two spheres of the same radius is a plane. However, the bisector of two spheres with different radii is a hyperboloid of two sheets or an ellipsoid (see the sidebar "Point-Sphere Bisector").

Freeform polynomial or rational primitives have bisectors significantly more complex than those for

linear or circular primitives. For example, in the plane the bisector of two cubic polynomial curves is an algebraic curve of degree 46 (see the sidebar "Degree of Planar Bisector Curves"). Moreover, the bisector curve is nonrational, in general. Because of these limitations, previous work has considered various approximation techniques for bisector curves in the plane. In particular, Farouki and Ramamurthy⁵ developed an approximation algorithm to reduce the approximation error within an arbitrary bound. The lack of exact rational form significantly hinders the use of bisectors in practical applications. The need to numerically approximate the bisector, typically using a piecewise linear curve, not only introduces a large error to the result but also causes an explosion in the amount of output data.

In some special cases the bisector has a simple closed form or a rational representation. Dutta and Hoffmann² considered the bisector of simple surfaces such as natural quadrics and toroidal surfaces in some special configurations. For these special types of surfaces and configurations, the bisector has a simple closed-form representation. Farouki and Johnstone⁶ showed that the bisector of a point and a rational curve in the same plane is a rational curve. Elber and Kim³ showed that the bisector of two rational space curves in R³ is a rational surface. Similarly, the bisector of a point and a rational space curve in R³ is a rational ruled surface.³

This article shows that the bisector of a point and a rational surface in R³ is also a rational surface. This result implies that the bisector of a sphere and a surface with a rational offset is also a rational surface. Even a simple rational bisector between two spheres and that between a point and a sphere have many important applications in practice. The bisector between a cube and a sphere consists of various surface patches, some of them are the bisectors between portions of the sphere and the corners of the cube. An application that uses the bisector of two spheres (of different radii) occurs in computing an optimal path for an airplane trying to avoid radar detection. Assuming each radar has different intensity, we can model the influence regions with spheres of different radii. The optimal path must lie on the bisector surface of the spheres.

Pottmann⁷ classified the class of all rational curves and surfaces with rational offsets, which includes the Pythagorean Hodograph (PH) curves as an important subclass of all polynomial curves with rational offsets.⁸ Simple surfaces (planes, spheres, cylinders, cones, and tori), Dupin cyclides, rational canal surfaces, and nondevelopable rational ruled surfaces all belong to this class of rational surfaces with rational offsets.⁹⁻¹¹ Consequently, our result applies to a wide variety of rational curves and surfaces for the construction of their rational bisector curves or surfaces with circles or spheres.

We define a canal surface as the sweep envelope surface of a moving sphere (possibly with a variable radius). Given a rational trajectory curve for the center of the sphere and a rational radius function of the sphere, Peternell and Pottmann¹⁰ present a surprising result—that the canal surface is also rational. The offset

Point-Sphere Bisector

Let $S_r(O) \subset \mathbb{R}^3$ be a sphere of radius *r* with its center at *O*. The bisector of a point $Q \in \mathbb{R}^3$ and a sphere $S_r(O)$ is either an ellipsoid or a hyperboloid of two sheets, depending on whether *Q* is contained in the interior or in the exterior of the sphere $S_r(O)$. Note that each point *P* on the bisector surface satisfies

 $\delta = d(P, S_r(O)) = ||P - Q||$ for some $\delta \ge 0$, where $d(P, S_r(O))$ is the distance between P and $S_r(O)$.

Case 1. *Q* lies inside the sphere $S_r(O)$, as in Figure A (see also Figure 2a).

In the configuration shown in Figure A, we have ||P - O|| + ||P - Q||

$$= (r - \delta) + \delta$$
$$= r$$

which means that, for each point *P* on the bisector, the sum of its distances to *O* and *Q* remains constant. Hence, the point *P* lies on an ellipsoid. Moreover, *O* and *Q* are the two focal points of the ellipsoid.

Case 2. *Q* lies outside the sphere $S_r(O)$, as in Figure B (see also Figure 2b).

In the configuration shown in Figure B, we have ||P - O|| - ||P - O||

 $= (r + \delta) - \delta$ = r

which means that, for each point *P* on the bisector, the difference of its distances to *O* and *Q* remains constant. Hence, the point *P* is a hyperboloid of two sheets. Moreover, *O* and *Q* are the two focal points of the hyperboloid.

Note that the bisector of two spheres $S_{r_1}(O_1)$ and $S_{r_2}(O_2)$ where $r_1 < r_2$, is identical to the bisector of the point O_1 and a sphere $S_{r_2-r_1}(O_2)$.



A *Q* lies inside the sphere $S_r(O)$.



B Q lies outside the sphere $S_r(Q)$.

Degree of Planar Bisector Curves

Given two planar polynomial curves $C_1(t) = (a(t), b(t))$ and $C_2(s) = (\alpha(s), \beta(s))$ of degrees m_1 and m_2 , respectively, the point P = (x, y) on the bisector curve of $C_1(t)$ and $C_2(s)$ satisfies the following three polynomial equations:³

$$\langle P - C_1(t), C'_1(t) \rangle = 0 \langle P - C_2(s), C'_2(s) \rangle = 0 \langle P - \frac{C_1(t) + C_2(s)}{2}, C_1(t) - C_2(s) \rangle = 0$$

 $\begin{array}{l} a'(t)x + b'(t)y = a(t)a'(t) + b(t)b'(t) & (1) \\ \alpha'(s)x + \beta'(s)y = \alpha(s)\alpha'(s) + \beta(s)\beta'(s) & (2) \\ e(s, t)x + f(s, t)y = g(s, t) & (3) \\ \text{where } e(s, t) = a(t) - \alpha(s), \ f(s, t) = b(t) - \beta(s), \ \text{and} \\ g(s, t) = 1/2 \ (a(t)^2 + b(t)^2 - \alpha(s)^2 - \beta(s)^2). \end{array}$

When we eliminate the variable t from Equations 1 and 3, we get an algebraic equation in three variables x, y, s:

(4)

(5)

$$F(x, y, s) = 0$$

By further eliminating the variable *s* from Equations 2 and 4, we get an algebraic equation in two variables *x* and *y*:

$$G(x, y) = 0$$

which represents the bisector curve of $C_1(t)$ and $C_2(s)$.

In the case of low degree curves $C_1(t)$ and $C_2(s)$, ($1 \le m_1 \le 3$ and $1 \le m_2 \le 6$), we have observed that the bisector curve of Equation 5 is an algebraic curve of degree $7m_1m_2 - 3(m_1 + m_2) + 1$,

which is also irreducible over rational coefficients. For two cubic curves, the degree is thus 46—too high to be useful in practice (see Elber and Kim⁴).

Rational Parameterization

$$\frac{\partial S(u, v)}{\partial u} = (a_{11}(u, v), a_{12}(u, v), a_{13}(u, v))$$

$$\frac{\partial S(u, v)}{\partial v} = (a_{21}(u, v), a_{22}(u, v), a_{23}(u, v))$$

$$S(u, v) - Q = (a_{31}(u, v), a_{32}(u, v), a_{33}(u, v))$$

$$\left\langle S(u, v), \frac{\partial S(u, v)}{\partial u} \right\rangle = b_1(u, v)$$

$$\left\langle S(u, v), \frac{\partial S(u, v)}{\partial v} \right\rangle = b_2(u, v)$$

$$\frac{|| S(u, v) ||^2 - || Q ||^2}{2} = b_3(u, v)$$

$$P(u, v) = (p_x(u, v), p_y(u, v), p_z(u, v))$$

then we can reformulate Equations 8 and 9 as follows:

$[a_{11}(u, v)]$	$a_{12}(u, v)$	$a_{13}(u, v)$	$p_x(u, v)$		$\int b_1(u, v)$
$a_{21}(u, v)$	$a_{22}(u, v)$	$a_{23}(u, v)$	$p_{y}(u, v)$	=	$b_2(u, v)$
$a_{31}(u, v)$	$a_{32}(u, v)$	$a_{33}(u, v)$	$p_z(u,v)$		$b_3(u, v)$

Note that $a_{i,j}(u, v)$ and $b_i(u, v)$, (i, j = 1, 2, 3) are all rational functions if S(u, v) is a rational surface. By Cramer's rule, we can obtain rational

of a canal surface is another canal surface, obtained by simply increasing the radius function by the offset distance and sweeping the enlarged sphere along the same trajectory. Consequently, rational canal surfaces (defined by rational trajectory curves and rational radius functions) remain closed under offset operation and admit rational representations for their bisector surfaces with arbitrary spheres.

After discussing the construction of a rational bisec-



1 (a) A bisector surface (in gray) of a point and a plane. A biquadratic surface is constructed that has the shape of a parabolic surface of revolution. (b) The bisector surface (in gray) of a point and a general bicubic surface.

parameterization of $p_x(u, v)$, $p_y(u, v)$, and $p_z(u, v)$ as follows:

p _x (u, v)	=	$ \begin{vmatrix} b_1(u, v) \\ b_2(u, v) \\ b_3(u, v) \end{vmatrix} $ $ \begin{vmatrix} a_{11}(u, v) \\ a_{21}(u, v) \\ a_{31}(u, v) \end{vmatrix} $	$ \begin{array}{c} a_{12}(u, v) \\ a_{22}(u, v) \\ a_{32}(u, v) \\ \end{array} \\ \begin{array}{c} a_{12}(u, v) \\ a_{22}(u, v) \\ a_{32}(u, v) \end{array} $	$\begin{array}{c} a_{13}(u, v) \\ a_{23}(u, v) \\ a_{33}(u, v) \\ \hline a_{13}(u, v) \\ a_{23}(u, v) \\ a_{33}(u, v) \end{array}$
р _у (и, v)	=	$\begin{vmatrix} a_{11}(u, v) \\ a_{21}(u, v) \\ a_{31}(u, v) \end{vmatrix}$ $\begin{vmatrix} a_{11}(u, v) \\ a_{21}(u, v) \\ a_{31}(u, v) \end{vmatrix}$	$b_{1}(u, v) \\ b_{2}(u, v) \\ b_{3}(u, v) \\ a_{12}(u, v) \\ a_{22}(u, v) \\ a_{32}(u, v)$	$\begin{array}{c} a_{13}(u, v) \\ a_{23}(u, v) \\ a_{33}(u, v) \\ \hline a_{13}(u, v) \\ a_{23}(u, v) \\ a_{33}(u, v) \end{array}$
p _z (u, v)	=	$\begin{vmatrix} a_{11}(u, v) \\ a_{21}(u, v) \\ a_{31}(u, v) \end{vmatrix}$ $\begin{vmatrix} a_{11}(u, v) \\ a_{21}(u, v) \\ a_{31}(u, v) \end{vmatrix}$	$\begin{array}{c} a_{12}(u, v) \\ a_{22}(u, v) \\ a_{32}(u, v) \\ a_{12}(u, v) \\ a_{22}(u, v) \\ a_{32}(u, v) \end{array}$	$\begin{array}{c} b_1(u, v) \\ b_2(u, v) \\ b_3(u, v) \\ a_{13}(u, v) \\ a_{23}(u, v) \\ a_{33}(u, v) \end{array}$

tor surface of a point and a rational surface, we extend the result to that for a rational bisector surface of a sphere and a rational surface with rational offsets. We created the proposed algorithm's implementation and all the examples presented in this article using Irit,¹² a solid modeling system developed at the Technion, Israel.

Bisector of a point and a surface

Let $Q \in \mathbb{R}^3$ be a fixed point and $S(u, v) \subset \mathbb{R}^3$ be a regular \mathbb{C}^1 -continuous parametric surface. Assume that P(u, v) is a bisector point of Q and S(u, v). Then we have

$$(P(u, v) - S(u, v)) || N(u, v)$$
(6)

$$||P(u,v) - Q|| = ||P(u,v) - S(u,v)||$$
(7)

where || denotes the parallel relationship, N(u, v) is the normal vector of S(u, v), and $|| \cdot ||$ denotes the length of a vector.

We can rewrite Equation 6 using the following two constraints:

$$\left\langle P(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle = 0$$

$$\left\langle P(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle = 0$$
(8)

whereas we can rewrite Equation 7 as follows:

$$\langle P(u,v) - Q, P(u,v) - Q \rangle = \langle P(u,v) - S(u,v), P(u,v) - S(u,v) \rangle$$



2 Two views of the bisector surface between a point and a sphere. In (a), the point lies inside the sphere and the bisector surface is an ellipsoid. In (b), the point lies outside the sphere and the bisector surface is a hyperboloid of two infinite sheets.

(9)





3 Views of the bisector surface between a point and a cone and that between a point and a cylinder. In (a), the point lies outside the cone forming two infinite sheets due to poles in the bisector's parameterization, while in (b), the point lies inside the cylinder.

or equivalently,

$$\left\langle P(u,v), S(u,v) - Q \right\rangle$$
$$= \frac{\left\langle S(u,v), S(u,v) \right\rangle - \left\langle Q, Q \right\rangle}{2}$$

The constraints in Equations 8 and 9 are all linear in P(u, v). Using Cramer's rule, we can solve these equations for P(u, v) and compute a rational surface representation of P(u, v) provided that S(u, v) is a rational surface (see the sidebar "Rational Parameterization").

Figure 1a shows the bisector surface of a point and a plane, which is a parabolic surface of revolution. Figure 1b considers the bisector surface of a point and a bicubic surface. The bisector surface has degree (11, 11).

Figures 2 through 4 present several examples of the bisector surface between a point and a simple surface. We represented the sphere using a rational biquadratic surface and the cone and cylinder using a rational NURBS (nonuniform rational B-spline) surface of degree (1, 2). We represented the torus as a biquadratic NURBS surface as well. The bisector of a point and a sphere turns out to be a rational parametric surface of degree (16, 16). The bisector of a point and a cone (or a cylinder) is of degree (16, 16). The bisector of a point and a torus is of degree (16, 16).

Bisector of a sphere and a surface with rational offsets

Let $S_d(Q)$ denote the sphere of radius d with its center at Q and a normal field pointing outside, and let $S_d(u, v)$ denote the offset surface of S(u, v) by an offset radius d:

$$S_d(u,v) = S(u,v) + d \frac{N(u,v)}{\|N(u,v)\|}$$

where N(u, v) is the normal vector field of S(u, v). Note that N(u, v) is the normal field of $S_d(u, v)$ as well, because the normal field is preserved under the offset operation.

Without loss of generality, we consider only the bisec-





4 Two views of the bisector surface between a point and a torus. In (a), the point lies at the center of the torus, while in (b), the point lies inside the torus.

tor points *P* along the positive normal directions: outside the sphere $S_d(Q)$ and along the oriented side of the normal direction N(u, v) of $S_d(u, v)$. Then, the bisector surface of the sphere $S_d(Q)$ and the offset surface $S_d(u, v)$ is the same as the bisector surface of the point *Q* and the given surface S(u, v). A point *P* at an equal distance *r* between $S_d(Q)$ and $S_d(u, v)$ lies at an equal distance r+d between *Q* and S(u, v), and vice versa, for all *d*. Similarly, the bisector surface of the sphere $S_d(Q)$ and the surface S(u, v) is the same as the bisector surface of the point *Q* and the offset surface $S_{-d}(u, v)$.

Assume that S(u, v) is a rational surface, which has a rational offset surface $S_r(u, v)$, for some $r \neq 0$. Then the offset surface $S_{-d}(u, v)$ is rational for all d, since we have

$$d(u,v) = S(u,v) - d \frac{N(u,v)}{\|N(u,v)\|}$$

= $S(u,v) - \frac{d}{r} r \frac{N(u,v)}{\|N(u,v)\|}$
= $S(u,v) - \frac{d}{r} (S_r(u,v) - S(u,v))$

 S_{-}

IEEE Computer Graphics and Applications

5 A view of the bisector surface between a sphere and a canal surface. We computed the bisector by offsetting the canal surface and computing the bisector between the center of the sphere and the offset of the canal surface.



where S(u, v) and $S_r(u, v) - S(u, v)$ clearly are rational. Consequently, the bisector between $S_d(Q)$ and S(u, v) is a rational surface if the surface S(u, v) has a rational offset surface.

Simple surfaces (such as planes, spheres, cylinders, cones, and tori) are all rational. Moreover, they have rational offsets, since the offsets of these simple surfaces are again simple surfaces of the same type. Therefore, we can compute the bisector surface between a sphere and a simple surface as a rational surface. Dupin cyclides and canal surfaces (defined by rational spine curves and rational radius functions) are also rational and closed under offset operation.^{9,10} Therefore, the bisector between a sphere and a surface (of these special types) also becomes a rational surface. Figure 5 shows an example of the bisector surface between a sphere and a canal surface.

Pottmann et al.¹¹ showed that all non-developable rational ruled surfaces also belong to the class of rational surfaces with rational offsets. Consequently, our result applies to a wide variety of rational curves or surfaces for constructing their rational bisector curves or surfaces with circles or spheres.

In Figure 5, notice that the bisector surface consists of three components, where two adjacent components meet at a self-intersection point of the bisector surface. Only the middle part belongs to the true bisector surface, since the other two components lie closer to the canal surface than to the sphere. We call the elimination of these redundant components *trimming*.

Given a point and an arbitrary surface, their trimmed bisector surface bounds a convex region that contains the point.⁶ Similarly, the trimmed bisector surface between a sphere and a surface also bounds a convex region that contains the sphere, since it is essentially the same as the bisector surface between the sphere center and an offset surface. Given a surface and a multilayer of concentric spheres, there exists a corresponding multilayer of convex trimmed bisector surfaces. When the given surface is a surface with rational offsets, we can generate a level set of convex trimmed rational bisector surfaces that propagate from a point. Moreover, each bisector surface has its rational parameterization inherited from that of the given rational surface. This capability lets us generate a new rational local coordinate system in the neighborhood of the given point.

In general, the trimmed bisector surface may consist of many subpatches bounded by the self-intersection curve of the untrimmed bisector. Consequently, developing an exact trimming procedure proves nontrivial, since it is difficult to determine a priori how many subpatches the trimmed bisector has. Based on the convexity of the region bounded by a trimmed bisector, we can

- develop a subdivision-based method that samples points on the bisector surface,
- construct supporting planes at the points,
- construct a convex polyhedron bounded by these planes, and
- trim out the bisector surface components that belong to the exterior of the convex polyhedron.

We can repeat this procedure for the surface regions where trimmings have occurred.

Conclusion

We have shown that the bisector between a point and a rational surface is a rational surface. This result implies that the bisector between a sphere and a rational surface with rational offsets is also a rational surface. Recent development of Pythagorean Hodograph space curves also accelerates the advance of surface design techniques for rational sweep surfaces with rational offsets. Consequently, the bisector construction scheme we have introduced in this article has much potential for practical applications.

Acknowledgments

We are thankful to Martin Peternell and Helmut Pottmann for providing us with their Maple program for constructing rational canal surfaces (and rational offsets also). We used this program for the example shown in Figure 5. This research was supported in part by Korea Science and Engineering Foundation (Kosef) under Grant 96-0100-01-01-2 and by the Fund for Promotion of Research at the Technion, Haifa, Israel.

References

- H. Persson, "NC Machining of Arbitrary-Shaped Pockets," Computer-Aided Design, Vol. 10, No. 3, May 1978, pp. 169-174.
- D. Dutta and C. Hoffmann, "On the Skeleton of Simple CSG Objects," ASME J. of Mechanical Design, Vol. 115, No. 1, Mar. 1993, pp. 87-94.
- G. Elber and M.-S. Kim, "The Bisector Surface of Freeform Rational Space Curves," *ACM Trans. on Graphics*, Vol. 17, No. 1, Jan. 1998, pp. 32-49.
- G. Elber and M.-S. Kim, "Bisector Curves of Planar Rational Curves," *Computer-Aided Design*, Vol. 30, No. 14, Dec. 1998, pp. 1089-1096.
- R. Farouki, and R. Ramamurthy, "Specified-Precision Computation of Curve/Curve Bisectors," *Int'l J. of Computational Geometry and Applications*, Vol. 8, Nos. 5-6, Oct.-Dec. 1998, pp. 599-617.

- R. Farouki and J. Johnstone, "The Bisector of a Point and a Plane Parametric Curve," *Computer-Aided Geometric Design*, Vol. 11, No. 2, Apr. 1994, pp. 117-151.
- H. Pottmann, "Rational Curves and Surfaces with Rational Offsets," *Computer-Aided Geometric Design*, Vol. 12, No. 2, Mar. 1995, pp. 175-192.
- R. Farouki and T. Sakkalis, "Pythagorean Hodographs," *IBM J. of Research and Development*, Vol. 34, No. 5, Sep. 1990, pp. 736-752.
- 9. D. Dutta, R. Martin, and M. Pratt, "Cyclides in Surface and Solid Modeling," *IEEE Computer Graphics and Applications*, Vol. 13, No. 1, Jan. 1993, pp. 53-59.
- M. Peternell and H. Pottmann, "Computing Rational Parameterizations of Canal Surfaces," J. of Symbolic Computation, Vol. 23, Nos. 2-3, 1997, pp. 255-266.
- H. Pottmann, W. Lü, and B. Ravani, "Rational Ruled Surfaces and Their Offsets," *Graphical Models and Image Processing*, Vol. 58, No. 6, Nov. 1996, pp. 544-552.
- 12. Irit 7.0 User's Manual, Technion, 1997, http://www.cs. technion.ac.il/~irit.



Gershon Elber is an associate professor in the Computer Science Department, Technion, Israel. His research interests span computeraided geometric designs and computer graphics. He received a BS in computer engineering and an MS in

computer science from the Technion in 1986 and 1987, respectively, and a PhD in computer science from the University of Utah in 1992. He is a member of the ACM and IEEE.



Myung-Soo Kim is an associate professor in the Department of Computer Engineering, Seoul National University, Korea. His research interests are in computer graphics and geometric modeling. Kim received BS and MS degrees in mathematics from

Seoul National University in 1980 and 1982, respectively. He continued his graduate studies at Purdue University, West Lafayette, Indiana, where he received an MS degree in applied mathematics in 1985 and MS and PhD degrees in computer science in 1987 and 1988, respectively.

Contact Elber at the Technion, Israel Institute of Technology, Dept. of Computer Science, Haifa 32000, Israel, email gershon@cs.technion.ac.il.

Contact Kim at Department of Computer Engineering, Seoul National University, Seoul 151-742, South Korea, e-mail mskim@comp.snu.ac.kr.





When Computers Went to Sea The Digitization of the United States Navy by David L. Boslaugh

This book explores the history of the United States Navy's secret development of codebreaking computers and their adaptation to solve critical fleet radar data handling problems. This is the only book written on the US Navy's initial application of shipboard digital computers to naval warfare.

Contents: Radar — New Eyes for

- the Fleet A Lingering Problem
- The Codebreaking Computers
- Conception of a New System
- Building a New System
- No Damned Computer is Going to Tell Me What to Do

New Horizons for Tactical

Computers • Twilight of the Analogs

255 pages. 7" x 10" Hardcover. May 1999. ISBN 0-7695-0024-2 — Catalog # BP00024 — \$28.00 Members / \$35.00 List

In Stock Today! Online Catalog

<u>http://computer.org</u> In the U.S. & Canada call

+1 800.CS.BOOKS

