Lecture Note #1 (4190.410)

Unit Quaternions

Quaternions were discovered by Sir William Hamilton in 1843 as a generalization of complex numbers. Instead of one imaginary unit i, three imaginary units i, j, k are used in quaternions:

$$\begin{gathered} 1 \cdot i = i, \quad 1 \cdot j = j, \quad 1 \cdot k = k, \qquad i^2 = j^2 = k^2 = -1, \\ i \cdot j = k, \quad j \cdot i = -k, \quad j \cdot k = i, \quad k \cdot j = -i, \qquad k \cdot i = j, \quad i \cdot k = -j. \end{gathered}$$

Each quaternion is represented as

$$q = w + xi + yj + zk,$$

where w, x, y, z are real numbers. We may represent the quaternion as a 4-tuple of real numbers: $q = (w, x, y, z)$.

For two quaternions: $q_1 = (w_1, x_1, y_1, z_1)$, $q_2 = (w_2, x_2, y_2, z_2)$, the quaternion addition and multiplication are defined as follows

$$\begin{aligned} q_1 + q_2 &= (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2), \\ q_1 \cdot q_2 &= (w_1 w_2 - \langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle, \\ &\qquad w_1 (x_2, y_2, z_2) + w_2 (x_1, y_1, z_1) + (x_1, y_1, z_1) \times (x_2, y_2, z_2)), \end{aligned}$$

where $\langle *, * \rangle$ means the inner product of two three-dimensional vectors.

Unit Quaternions and 3D Rotations

Unit quaternions are closely related to three-dimensional rotations. Given a unit quaternion $q = (w, x, y, z) \in S^3$, $w^2 + x^2 + y^2 + z^2 = 1$, we can represent it as follows

$$q = (w, x, y, z) = (\cos\theta, \sin\theta(a, b, c)),$$

where

$$\begin{aligned} (a, b, c) &= \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{w}\right). \end{aligned}$$

The unit quaternion $q = (\cos\theta, \sin\theta(a, b, c)) \in S^3$ represents the rotation by angle 2θ about $(a, b, c) \in S^2$. For any three-dimensional point $(\alpha, \beta, \gamma) \in R^3$, we can show that

$$(\cos\theta, \sin\theta(a, b, c)) \cdot (0, \alpha, \beta, \gamma) \cdot (\cos\theta, -\sin\theta(a, b, c)) = (0, \bar{\alpha}, \bar{\beta}, \bar{\gamma}),$$

which is the result of rotating (α, β, γ) by angle 2θ about the axis parallel to (a, b, c).

The Rotation Matrix

$$
\begin{aligned}
& (w, x, y, z) \cdot (0, \alpha, \beta, \gamma) \cdot (w, -x, -y, -z) \\
= \; & (-(x\alpha + y\beta + z\gamma), w(\alpha, \beta, \gamma) + (x, y, z) \times (\alpha, \beta, \gamma)) \cdot (w, -x, -y, -z) \\
= \; & (-w(x\alpha + y\beta + z\gamma) + w(\alpha x + \beta y + \gamma z), \\
& \quad (x\alpha + y\beta + z\gamma)(x, y, z) + w^2(\alpha, \beta, \gamma) \\
& \quad +w(x, y, z) \times (\alpha, \beta, \gamma) - w(\alpha, \beta, \gamma) \times (x, y, z) \\
& \quad -((x, y, z) \times (\alpha, \beta, \gamma)) \times (x, y, z)) \\
= \; & (0, (x^2\alpha + xy\beta + xz\gamma, xy\alpha + y^2\beta + yz\gamma, xz\alpha + yz\beta + z^2\gamma) \\
& \quad (w^2\alpha, \qquad w^2\beta, \qquad w^2\gamma) \\
& \quad (2wy\gamma - 2wz\beta, \quad 2wz\alpha - 2wx\gamma, \quad 2wx\beta - 2wy\alpha) \\
& \quad (xy\beta + xz\gamma - z^2\alpha - y^2\alpha, \\
& \qquad\qquad xy\alpha + yz\gamma - x^2\beta - z^2\beta, \\
& \qquad\qquad\qquad xz\alpha + yz\beta - x^2\gamma - y^2\gamma)) \\
= \; & (0, (x^2 + w^2 - y^2 - z^2)\alpha + (2xy - 2wz)\beta + (2xz + 2wy)\gamma, \\
& \quad (2xy + 2wz)\alpha + (y^2 + w^2 - x^2 - z^2)\beta + (2yz - 2wx)\gamma, \\
& \quad (2xz - 2wy)\alpha + (2yz + 2wx)\beta + (w^2 + z^2 - x^2 - y^2)\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}
&= \begin{bmatrix} x^2 + w^2 - y^2 - z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & y^2 + w^2 - x^2 - z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & w^2 + z^2 - x^2 - y^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}
\end{aligned}
$$

For each unit quaternion $q = (w, x, y, z) \in S^3$, let R_q denote the above 3×3 matrix. Then one can check that R_q is a three-dimensional rotational matrix, i.e., $R_q \in SO(3)$:

1. Each row is a unit vector, and each column is a unit vector.
2. Rows are mutually orthogonal each other, and columns are mutually orthogonal each other.
3. The determinant of R_q is 1.

Remark:

1. $R_{-q} = R_q$.
2. If $q_1, q_2 \in S^3$, then $q_2 \cdot q_1 \in S^3$.
3. $R_{q_2} R_{q_1} = R_{q_2 \cdot q_1}$.

Quaternion Calculus

Given a unit quaternion curve $q(t) \in S^3$, its derivative $q'(t)$ is given in the following form:

$$q'(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$.

To show this, we define

$$\gamma(h) = q(t+h) \cdot \overline{q(t)},$$

where $\overline{q(t)} = (w(t), -x(t), -y(t), -z(t)) \in S^3$ is the conjugate of $q(t)$. Note that $\gamma(h) \in S^3$ and

$$\gamma(0) = q(t) \cdot \overline{q(t)} = (1, 0, 0, 0).$$

(For each unit quaternion q, its conjugate $\overline{q}$ is also the inverse of q with respect to the quaternion multiplication. We can also show that $\overline{q_1 q_2} = \overline{q_2 q_1}$.) Thus $\gamma'(0)$ is orthogonal to $(1, 0, 0, 0)$ in R^4. Moreover, consider the following relation

$$\begin{aligned} q(t+h) &= q(t+h) \cdot \overline{q(t)} \cdot q(t) \\ &= \gamma(h) \cdot q(t). \end{aligned}$$

Differentiating by h, we have

$$q'(t+h) = \gamma'(h) \cdot q(t).$$

Setting $h = 0$ in the above equation, we finally get

$$q'(t) = \gamma'(0) \cdot q(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$. (Note that $\gamma'(0)$ is orthogonal to $(1, 0, 0, 0)$ and it is also dependent on t.)

Angular Velocity

Given a point $p \in R^3$ and a continuous rotation $R_{q(t)}$ implied by a unit quaternion curve $q(t) \in S^3$, the rotated point $p(t) = R_{q(t)}(p)$ is contained in a sphere with radius $\|p\|$ and center $(0, 0, 0)$:

$$(0, p(t)) = q(t) \cdot (0, p) \cdot \overline{q(t)}.$$

Differentiating the above, we get

$$\begin{aligned} (0, p'(t)) &= q'(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{q'(t)} \\ &= (0, v(t)) \cdot q(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{(0, v(t)) \cdot q(t)} \\ &= (0, v(t)) \cdot q(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{q(t)} \cdot \overline{(0, v(t))} \\ &= (0, v(t)) \cdot (0, p(t)) + (0, p(t)) \cdot (0, -v(t)) \\ &= (0, v(t)) \cdot (0, p(t)) + (0, v(t)) \cdot (0, p(t)) \\ &= (0, 2v(t)) \cdot (0, p(t)). \end{aligned}$$

This implies that the angular velocity $\omega(t)$ of the rotation $R_{q(t)}$ is given as follows

$$\omega(t) = 2v(t).$$