Lecture Note (4190.410)

Unit Quaternions

Quaternions were discovered by Sir William Hamilton in 1843 as a generalization of complex numbers. Instead of one imaginary unit i, three imaginary units i, j, k are used in quaternions:

$$1 \cdot i = i, \quad 1 \cdot j = j, \quad 1 \cdot k = k, \qquad i^2 = j^2 = k^2 = -1,$$
$$i \cdot j = k, \quad j \cdot i = -k, \quad j \cdot k = i, \quad k \cdot j = -i, \qquad k \cdot i = j, \quad i \cdot k = -j.$$

Each quaternion is represented as

$$q = w + xi + yj + zk,$$

where w, x, y, z are real numbers. We may represent the quaternion as a 4-tuple of real numbers: $q = (w, x, y, z)$.

For two quaternions: $q_1 = (w_1, x_1, y_1, z_1)$, $q_2 = (w_2, x_2, y_2, z_2)$, the quaternion addition and multiplication are defined as follows

$$\begin{aligned} q_1 + q_2 &= (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2), \\ q_1 \cdot q_2 &= (w_1 w_2 - \langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle, \\ &\qquad w_1(x_2, y_2, z_2) + w_2(x_1, y_1, z_1) + (x_1, y_1, z_1) \times (x_2, y_2, z_2)), \end{aligned}$$

where $\langle *, * \rangle$ means the inner product of two three-dimensional vectors.

Unit Quaternions and 3D Rotations

Unit quaternions are closely related to three-dimensional rotations. Given a unit quaternion $q = (w, x, y, z) \in S^3$, $w^2 + x^2 + y^2 + z^2 = 1$, we can represent it as follows

$$q = (w, x, y, z) = (\cos\theta, \sin\theta(a, b, c)),$$

where

$$\begin{aligned} (a, b, c) &= \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{w}\right). \end{aligned}$$

The unit quaternion $q = (\cos\theta, \sin\theta(a, b, c)) \in S^3$ represents the rotation by angle 2θ about $(a, b, c) \in S^2$. For any three-dimensional point $(\alpha, \beta, \gamma) \in R^3$, we can show that

$$(\cos\theta, \sin\theta(a, b, c)) \cdot (0, \alpha, \beta, \gamma) \cdot (\cos\theta, -\sin\theta(a, b, c)) = (0, \bar{\alpha}, \bar{\beta}, \bar{\gamma}),$$

which is the result of rotating (α, β, γ) by angle 2θ about the axis parallel to (a, b, c).

The Rotation Matrix

$$
\begin{aligned}
& (w,x,y,z)\cdot(0,\alpha,\beta,\gamma)\cdot(w,-x,-y,-z) \\
= \; & (-(x\alpha+y\beta+z\gamma), w(\alpha,\beta,\gamma)+(x,y,z)\times(\alpha,\beta,\gamma))\cdot(w,-x,-y,-z) \\
= \; & (-w(x\alpha+y\beta+z\gamma)+w(\alpha x+\beta y+\gamma z), \\
& \quad (x\alpha+y\beta+z\gamma)(x,y,z)+w^2(\alpha,\beta,\gamma) \\
& \quad +w(x,y,z)\times(\alpha,\beta,\gamma)-w(\alpha,\beta,\gamma)\times(x,y,z) \\
& \quad -((x,y,z)\times(\alpha,\beta,\gamma))\times(x,y,z)) \\
= \; & (0,(x^2\alpha+xy\beta+xz\gamma, xy\alpha+y^2\beta+yz\gamma, xz\alpha+yz\beta+z^2\gamma) \\
& \quad (w^2\alpha, \qquad w^2\beta, \qquad w^2\gamma) \\
& \quad (2wy\gamma-2wz\beta, \quad 2wz\alpha-2wx\gamma, \quad 2wx\beta-2wy\alpha) \\
& \quad (xy\beta+xz\gamma-z^2\alpha-y^2\alpha, \\
& \qquad\qquad xy\alpha+yz\gamma-x^2\beta-z^2\beta, \\
& \qquad\qquad\qquad xz\alpha+yz\beta-x^2\gamma-y^2\gamma)) \\
= \; & (0,(x^2+w^2-y^2-z^2)\alpha+(2xy-2wz)\beta+(2xz+2wy)\gamma, \\
& \quad (2xy+2wz)\alpha+(y^2+w^2-x^2-z^2)\beta+(2yz-2wx)\gamma, \\
& \quad (2xz-2wy)\alpha+(2yz+2wx)\beta+(w^2+z^2-x^2-y^2)\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix}\bar\alpha\\ \bar\beta\\ \bar\gamma\end{bmatrix}
&= \begin{bmatrix} x^2+w^2-y^2-z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & y^2+w^2-x^2-z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & w^2+z^2-x^2-y^2 \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} \\
&= \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}
\end{aligned}
$$

For each unit quaternion $q=(w,x,y,z)\in S^3$, let R_q denote the above 3×3 matrix. Then one can check that R_q is a three-dimensional rotational matrix, i.e., $R_q\in SO(3)$:

1. Each row is a unit vector, and each column is a unit vector.
2. Rows are mutually orthogonal each other, and columns are mutually orthogonal each other.
3. The determinant of R_q is 1.

Remark:

1. $R_{-q}=R_q$.
2. If $q_1,q_2\in S^3$, then $q_2\cdot q_1\in S^3$.
3. $R_{q_2}R_{q_1}=R_{q_2\cdot q_1}$.