Projection in 3D

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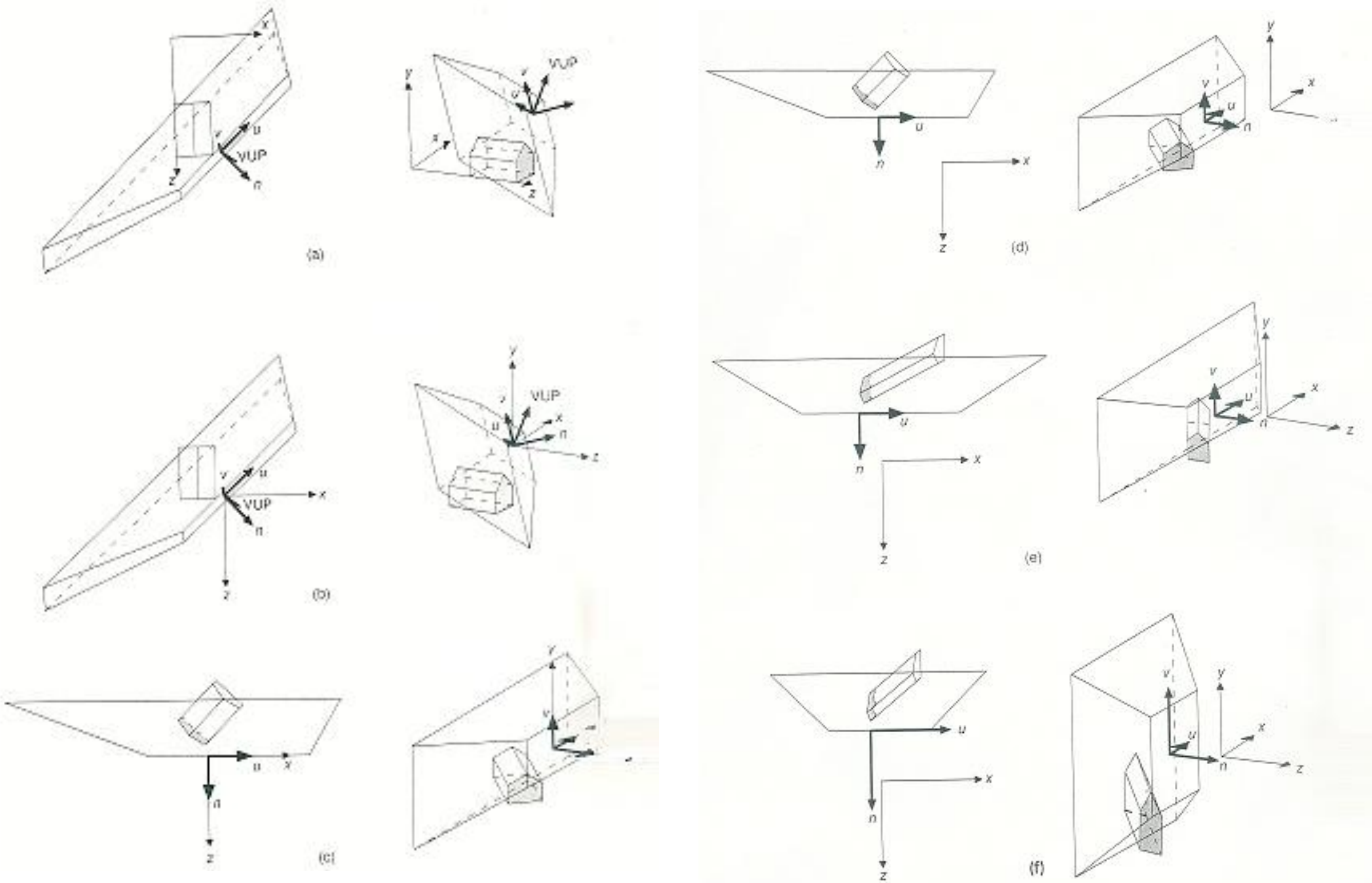
Conventional Way in Graphics



Plane in Space

$$\begin{aligned} \mathbf{n} &= (a, b, c) \\ \mathbf{x} - \mathbf{x}_0 &= (x - x_0, y - y_0, z - z_0) \end{aligned}$$

$$\begin{aligned} 0 &= \langle \mathbf{n}, \mathbf{x} - \mathbf{x}_0 \rangle \\ 0 &= a(x - x_0) + b(y - y_0) + c(z - z_0) \\ 0 &= ax + by + cz - ax_0 - by_0 - cz_0 \\ 0 &= ax + by + cz + d \end{aligned}$$

Plane by Three Points

$$\begin{aligned}
\hat{\mathbf{n}} &= (a, b, c, d) \\
\hat{\mathbf{x}}_0 &= (x_0, y_0, z_0, 1) \\
\hat{\mathbf{x}}_1 &= (x_1, y_1, z_1, 1) \\
\hat{\mathbf{x}}_2 &= (x_2, y_2, z_2, 1)
\end{aligned}$$

$$\begin{aligned}
ax_0 + by_0 + cz_0 + d \cdot 1 &= 0 \\
ax_1 + by_1 + cz_1 + d \cdot 1 &= 0 \\
ax_2 + by_2 + cz_2 + d \cdot 1 &= 0
\end{aligned}$$

Plane by Three Points

$$ax_0 + by_0 + cz_0 + d \cdot 1 = 0$$
$$ax_1 + by_1 + cz_1 + d \cdot 1 = 0$$
$$ax_2 + by_2 + cz_2 + d \cdot 1 = 0$$

$$\langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}}_0 \rangle = 0$$
$$\langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}}_1 \rangle = 0$$
$$\langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}}_2 \rangle = 0$$
$$\widehat{\mathbf{n}} = \widehat{\mathbf{x}}_0 \wedge \widehat{\mathbf{x}}_1 \wedge \widehat{\mathbf{x}}_2$$

Wedge Product

$$\hat{\mathbf{n}} = \hat{\mathbf{x}}_0 \wedge \hat{\mathbf{x}}_1 \wedge \hat{\mathbf{x}}_2$$

$$= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{bmatrix}$$

Point by Three Planes

$$\hat{\mathbf{x}} = (x, y, z, w)$$

$$\hat{\mathbf{n}}_0 = (a_0, b_0, c_0, d_0)$$

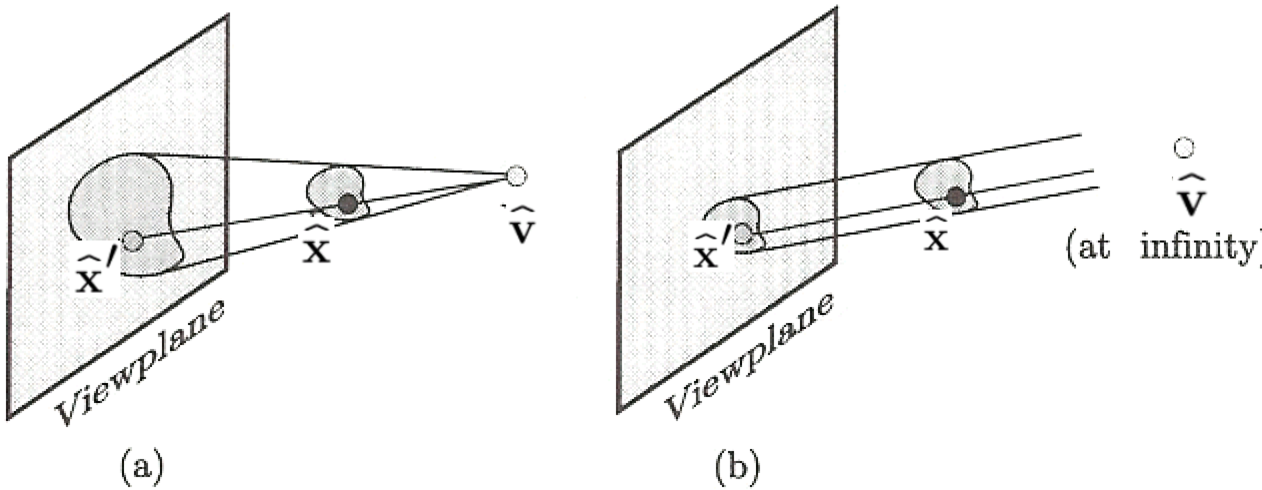
$$\hat{\mathbf{n}}_1 = (a_1, b_1, c_1, d_1)$$

$$\hat{\mathbf{n}}_2 = (a_2, b_2, c_2, d_2)$$

$$\hat{\mathbf{x}} = \hat{\mathbf{n}}_0 \wedge \hat{\mathbf{n}}_1 \wedge \hat{\mathbf{n}}_2$$

Projection in 3D

$$\widehat{\mathbf{x}}' = \langle \widehat{\mathbf{n}}, \widehat{\mathbf{v}} \rangle \, \widehat{\mathbf{x}} - \widehat{\mathbf{v}} \, \langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}} \rangle$$



Perspective and parallel three-dimensional projections

Projection in 3D

$$\hat{\mathbf{x}}' = \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \hat{\mathbf{x}} - \hat{\mathbf{v}} \langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle$$

(Case I): $\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle = 0$,

$$\hat{\mathbf{x}}' = \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \hat{\mathbf{x}} = \hat{\mathbf{x}}$$

(Case II): $\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle \neq 0$,

$$\begin{aligned} \hat{\mathbf{x}}' &= \alpha \hat{\mathbf{x}} + \beta \hat{\mathbf{v}} \\ 0 &= \langle \hat{\mathbf{n}}, \hat{\mathbf{x}}' \rangle \\ 0 &= \langle \hat{\mathbf{n}}, \alpha \hat{\mathbf{x}} + \beta \hat{\mathbf{v}} \rangle \end{aligned}$$

Projection in 3D

$$\hat{\mathbf{x}}' = \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \hat{\mathbf{x}} - \hat{\mathbf{v}} \langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle$$

(Case II): $\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle \neq 0$,

$$\begin{aligned}
\hat{\mathbf{x}}' &= \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{v}} \\
0 &= \langle \hat{\mathbf{n}}, \hat{\mathbf{x}}' \rangle \\
0 &= \langle \hat{\mathbf{n}}, \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{v}} \rangle \\
0 &= \alpha \langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle + \beta \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \\
\alpha &= -\beta \frac{\langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle}{\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle}
\end{aligned}$$

Projection in 3D

$$\hat{\mathbf{x}}' = \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \hat{\mathbf{x}} - \hat{\mathbf{v}} \langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle$$

(Case II): $\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle \neq 0$,

$$\alpha = -\beta \frac{\langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle}{\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle}$$

$$\begin{aligned} \hat{\mathbf{x}}' &= -\beta \frac{\langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle}{\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle} \hat{\mathbf{x}} + \beta \hat{\mathbf{v}} \\ &= \frac{\langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle}{\langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle} \hat{\mathbf{x}} - \hat{\mathbf{v}} \\ &= \langle \hat{\mathbf{n}}, \hat{\mathbf{v}} \rangle \hat{\mathbf{x}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{x}} \rangle \hat{\mathbf{v}} \end{aligned}$$

Perspective Projection in 3D

$$\widehat{\mathbf{x}}' = \langle \widehat{\mathbf{n}}, \widehat{\mathbf{v}} \rangle \, \widehat{\mathbf{x}} - \widehat{\mathbf{v}} \, \langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}} \rangle$$

$$\begin{bmatrix} x'w' \\ y'w' \\ z'w' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$- \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Parallel Projection in 3D

$$\widehat{\mathbf{x}}' = \langle \widehat{\mathbf{n}}, \widehat{\mathbf{v}} \rangle \, \widehat{\mathbf{x}} - \widehat{\mathbf{v}} \, \langle \widehat{\mathbf{n}}, \widehat{\mathbf{x}} \rangle$$

$$\begin{bmatrix} x'w' \\ y'w' \\ z'w' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$- \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Viewing Transform in 3D

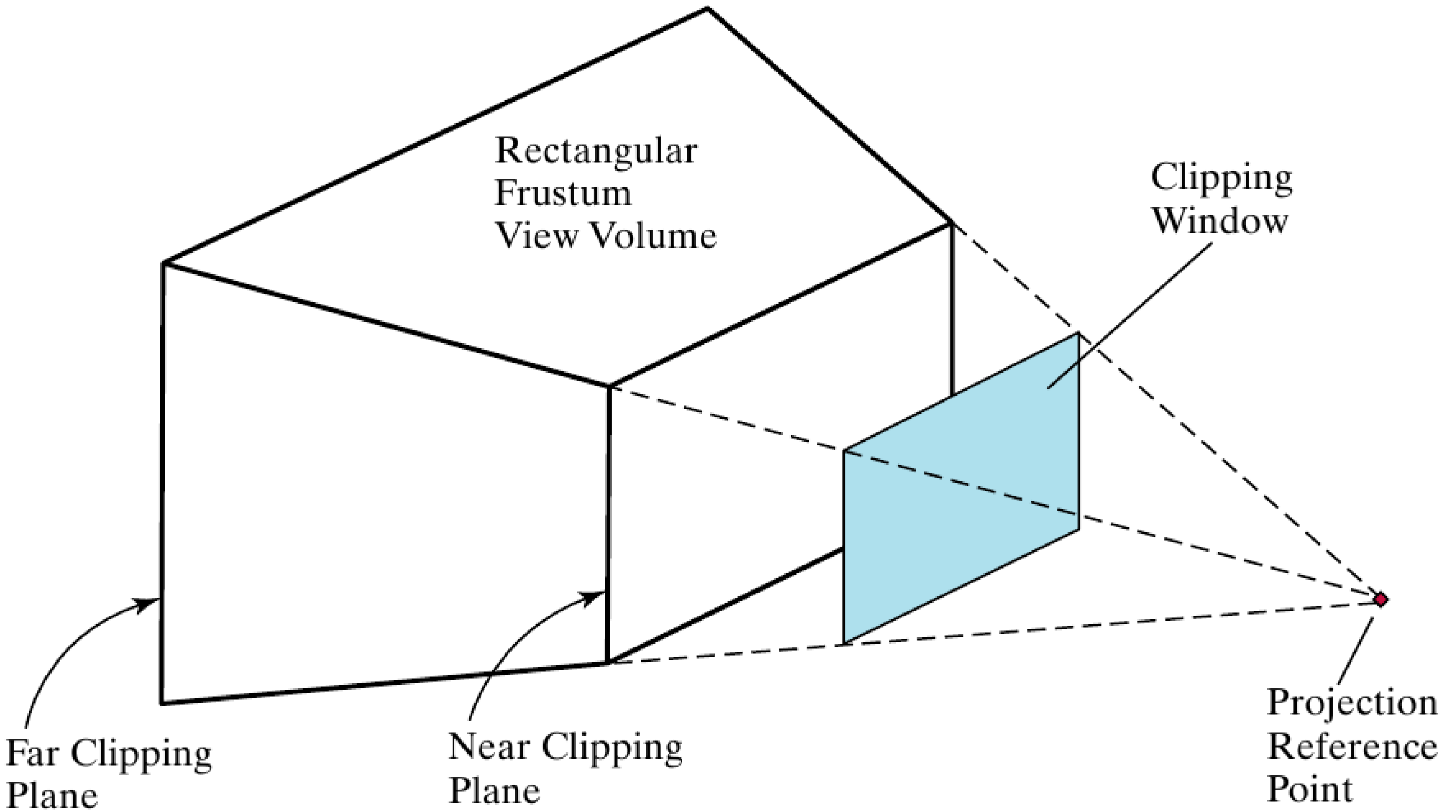
$$\begin{aligned}\mathbf{x} &= (x_0, y_0, z_0)\\ \mathbf{u} &= (u_1, u_2, u_3)\\ \mathbf{v} &= (v_1, v_2, v_3)\\ \mathbf{n} &= (n_1, n_2, n_3)\end{aligned}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{n} \rangle = \langle \mathbf{v}, \mathbf{n} \rangle = 0$$

$$\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{n}\| = 1$$

$$\begin{bmatrix} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Depth Transform in 3D



Depth Transform in 3D

$$ax + by + cz + d_0 = 0 : \quad \text{Near}$$

$$ax + by + cz + d_1 = 0 : \quad \text{Far}$$

$$ax + by + cz + d = 0 : \quad (x, y, z)$$

How can you check whether the point (x, y, z) is between the two planes?

How can you formulate the relative depth as an affine transformation?

Depth Transform in 3D

$$ax + by + cz + d_0 = 0 : \quad \text{Near}$$

$$ax + by + cz + d_1 = 0 : \quad \text{Far}$$

$$ax + by + cz + d = 0 : \quad (x, y, z)$$

(x, y, z) is between the two planes
if and only if

$$(d - d_0)(d - d_1) \leq 0$$

Depth Transform in 3D

$$ax + by + cz + d_0 = 0 : \quad \text{Near}$$

$$ax + by + cz + d_1 = 0 : \quad \text{Far}$$

$$ax + by + cz + d = 0 : \quad (x, y, z)$$

$$\begin{bmatrix} a & b & c & d_0 \\ 0 & 0 & 0 & d_0 - d_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + cz + d_0 \\ d_0 - d_1 \end{bmatrix}$$

$$= \begin{bmatrix} d_0 - d \\ d_0 - d_1 \end{bmatrix} = \frac{d_0 - d}{d_0 - d_1}$$