Unit Quaternion and 3D Rotations

Myung-Soo Kim
Seoul National University

Quaternions

$$1 \cdot i = i, \quad 1 \cdot j = j, \quad 1 \cdot k = k,$$
$$i^2 = j^2 = k^2 = -1,$$

$$i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j,$$
$$j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j.$$

$$q = w + xi + yj + zk,$$

where w, x, y, z are real numbers.

Addition and Multiplication

For $q_1 = (w_1, x_1, y_1, z_1)$, $q_2 = (w_2, x_2, y_2, z_2)$,

$$
\begin{aligned}
q_1 + q_2 &= (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2), \\
q_1 \cdot q_2 &= (w_1 w_2 - \langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle, \\
&\quad w_1(x_2, y_2, z_2) + w_2(x_1, y_1, z_1) \\
&\quad + (x_1, y_1, z_1) \times (x_2, y_2, z_2)),
\end{aligned}
$$

where $\langle *, * \rangle$ means the inner product.

Unit Quaternion

For $q = (w, x, y, z) \in S^3$, $w^2 + x^2 + y^2 + z^2 = 1$,

$$q = (w, x, y, z) = (\cos\theta, \sin\theta(a, b, c)) \in S^3,$$

where

$$\begin{aligned} (a, b, c) &= \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \in S^2, \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{w}\right). \end{aligned}$$

3D Rotation

For a 3D point $(\alpha, \beta, \gamma) \in R^3$,
a unit quaternion $q = (w, x, y, z) \in S^3$,
and its conjugate $\overline{q} = (w, -x, -y, -z) \in S^3$,

$$q \cdot (0, \alpha, \beta, \gamma) \cdot \overline{q} = (0, \bar{\alpha}, \bar{\beta}, \bar{\gamma}),$$

is the result of rotating (α, β, γ) by angle 2θ about the axis parallel to (a, b, c).

The 3D Rotation Matrix

$$\begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}$$

1. Each row is a unit vector, and each column is a unit vector.
2. Rows are mutually orthogonal each other, and columns are mutually orthogonal each other.
3. The determinant of R_q is 1.

Some Properties

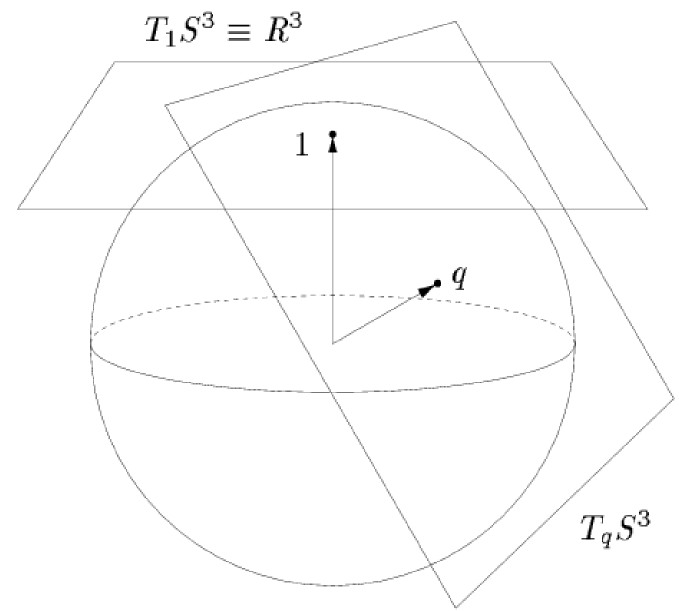
1. $R_{-q} = R_q$.
2. If $q_1, q_2 \in S^3$, then $q_2 \cdot q_1 \in S^3$.
3. $R_{q_2} R_{q_1} = R_{q_2 \cdot q_1}$.

Quaternion Calculus

Given $q(t) \in S^3$,

$$q'(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$.



Angular Velocity

The rotated point $p(t) = R_{q(t)}(p)$ is in a sphere with radius $\|p\|$ and center $(0, 0, 0)$:

$$(0, p(t)) = q(t) \cdot (0, p) \cdot \overline{q(t)}.$$

Differentiating the above, we get

$$\begin{aligned} (0, p'(t)) &= (0, 2v(t)) \cdot (0, p(t)) \\ &= (0, 2v(t) \times p(t)), \end{aligned}$$

which means $\omega(t) = 2v(t)$.