Old Exam Problems

1. (15 points) Show that
$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$$

2. (15 points) Consider a cubic Bézier curve with four control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$:
$$C(t) = \sum_{i=0}^{3} \mathbf{p}_i B_i(t),$$
where
$$B_i(t) = \left(\begin{array}{c} 3 \\ i \end{array} \right) (1-t)^{3-i} t^i.$$
Show that this curve is translation invariant; that is, the control points of $C(t) + \mathbf{q}$ are
$$\mathbf{p}_0 + \mathbf{q},\ \mathbf{p}_1 + \mathbf{q},\ \mathbf{p}_2 + \mathbf{q},\ \mathbf{p}_3 + \mathbf{q}.$$

3. (20 points) Consider a cubic Bézier curve $C(t)$, $0 \le t \le 1$, with control points: $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$. Represent the other curve segment of $C(t)$, $1 \le t \le 3$, as a cubic Bézier curve by constructing four control points $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2$, and $\mathbf{q}_3$.

4. (15 points)
   (a) (5 points) Represent $C(t) = (2t, 8t^3 - 6t + 1)$ as a cubic Bézier curve.
   (b) (5 points)Subdivide this curve into two subsegments $C_1(u)$ and $C_2(v)$.
   (c) (5 points) Can you guarantee that each of the two curves $C_1(u)$ and $C_2(v)$ intersects the x-axis exactly at one point? Justify your answer.

5. (10 points) A Bézier curve $B(t)$ is given by the four control points $\mathbf{b}_0 = (0.3, 0.1)$, $\mathbf{b}_1 = (0.9, 0.6)$, $\mathbf{b}_2 = (1.3, -0.1)$, $\mathbf{b}_3 = (0.7, -0.4)$. Use the de Casteljau algorithm to compute the control points defining B_{left} and B_{right} obtained by subdividing $B(t)$ at $t = \frac{1}{3}$.

6. (15 points) Let $\mathbf{x}(t),\ 0 \le t \le 1$, be a cubic Bézier curve given by four control points:
$$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$
Consider another cubic polynomial curve $\mathbf{y}(u),\ 0 \le u \le 1$, which is an extension of the curve $\mathbf{x}(t)$ over a longer parameter interval $[-1, 2]$; namely, $\mathbf{y}(u) = \mathbf{x}(3u - 1)$, $0 \le u \le 1$. What is the representation of $\mathbf{y}(u), 0 \le u \le 1$, as a cubic Bézier curve? What are the four control points $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ of $\mathbf{y}(u)$?

7. (15 points) What are the control points $\mathbf{b}_{ij}$ for a bicubic Bézier patch defining the function graph $(x, y, x^2 + y^2)^T$ over the rectangular domain $[2, 5] \times [1, 7]$?