Chap 4. Determinants

4.2 Ten Properties

① $\det(I)=1$

② Sign changes when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb-da = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

③ Linearly dependent on each row/column.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

[Note: $\det(B+C) \neq \det(B)+\det(C)$
$\det(tA) = t^n \det(A) \neq t\det(A)$ if $n \neq 1$]

④ $\det A = 0$ if two rows are equal.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab-ba=0$$

⑤ Subtracting a multiple of one row from another row leaves the same determinant.

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

⑥ $\det A = 0$ if there is a row of zeros.

⑦ $\det A = a_{11}a_{22}\cdots a_{nn}$ for a triangular matrix.

⑧ $\det A = 0 \Leftrightarrow A$ is singular
$\det A \neq 0 \Leftrightarrow A$ is invertible

⑨ $\det(AB) = \det A \cdot \det B$, $\det A^{-1} = 1/\det A$

⑩ $\det A^T = \det A$.

∵ $A = LDU \Rightarrow A^T = U^T D^T L^T \Rightarrow \det(A^T) = \det(\overset{\overset{D}{\|}}{D^T}) = \det(A)$

4.4 Applications of Determinants

o. The Solution of $A\mathbf{x} = \mathbf{b}$ (Cramer's Rule)

$$x_j = \frac{\det B_j}{\det A}, \quad \text{where } B_j = \begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & & & & \\ a_{n1} & & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$

o. The Volume of a Box

- When the edges are perpendicular and the box is rectangular, the volume is $l_1 l_2 \cdots l_n$.

$$AA^T = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \text{r} & & \text{r} \\ \text{o} & \cdots & \text{o} \\ \text{w} & & \text{w} \\ 1 & & n \end{bmatrix} = \begin{bmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{bmatrix}$$

$$l_1^2 l_2^2 \cdots l_n^2 = \det(AA^T) = (\det A)(\det A^T) = (\det A)^2$$

- When the angles are not $90°$, we can change the parallelogram to a rectangle without changing the volume.

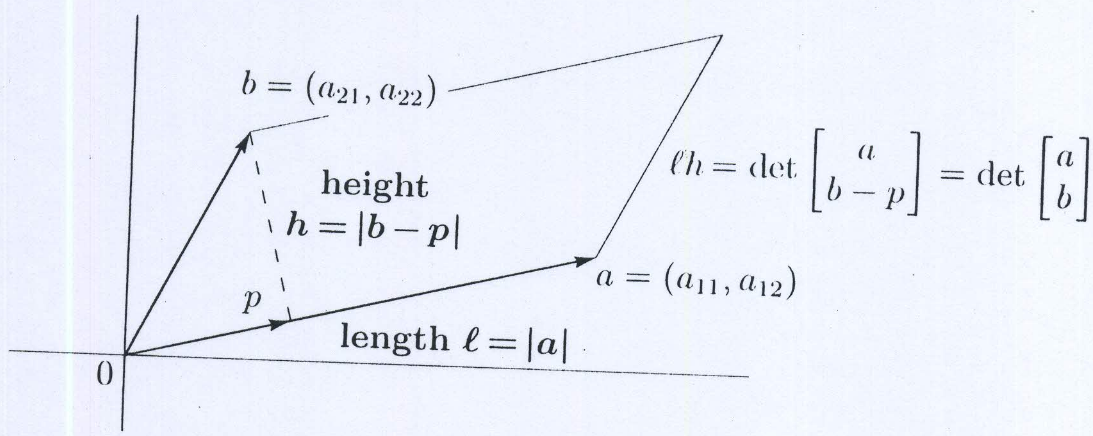

Figure 4.2 Volume (area) of the parallelogram $= \ell$ times $h = |\det A|$.

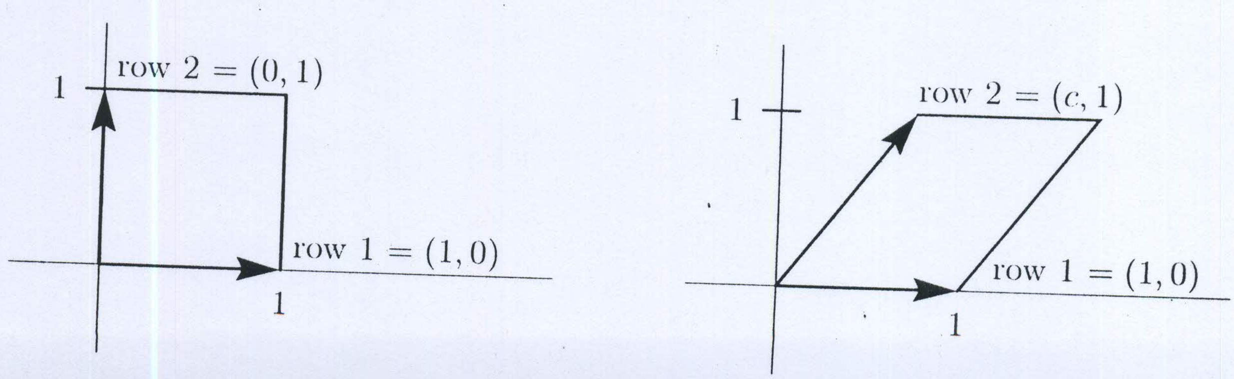

Figure 4.3 The areas of a unit square and a unit parallelogram are both 1.