4. (15 points)

(a) (6 points) Compute the projection matrices $P_i = \mathbf{a}_i\mathbf{a}_i^T/\mathbf{a}_i^T\mathbf{a}_i$, $i = 1, 2, 3$, onto the lines $\mathbf{a}_1 = (-1, 2, 2)$, $\mathbf{a}_2 = (2, 2, -1)$, and $\mathbf{a}_3 = (2, -1, 2)$.

(b) (3 points) Project $\mathbf{b} = (1, 0, 0)$ to $\mathbf{p}_i$ on the lines through $\mathbf{a}_i$, $i = 1, 2, 3$.

(c) (6 points) Verify that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{b}$ and $P_1 + P_2 + P_3 = I$.

(a) $$P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \quad P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

(b) $$\mathbb{P}_1 = \left(\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9}\right), \quad \mathbb{P}_2 = \left(\frac{4}{9}, \frac{4}{9}, -\frac{2}{9}\right)$$

$$\mathbb{P}_3 = \left(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right)$$

(c) $$\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 = \left(\frac{9}{9}, 0, 0\right) = \mathbb{b}$$

$$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

5. (20 points) We want to fit a plane $z = C + Dx + Ey$ to the four points:

$$\begin{array}{llll} z=3 \quad \text{at} \quad x=1,\ y=1; & z=6 \quad \text{at} \quad x=0,\ y=3 \\ z=5 \quad \text{at} \quad x=2,\ y=1; & z=0 \quad \text{at} \quad x=0,\ y=0 \end{array}$$

(a) (10 points) Find 4 equations in 3 unknowns to pass a plane through the points (if there is such a plane).

(b) (10 points) Find 3 equations in 3 unknowns for the best least-squares solution.

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 3 & 5 \\ 3 & 6 & 3 \\ 5 & 3 & 11 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 16 \end{bmatrix}$$

6. (10 points) True or false (give an example in either case):

(a) (5 points) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

(b) (5 points) If a 3×2 matrix Q has orthonormal columns, then $\|Q\mathbf{x}\|$ always equals $\|\mathbf{x}\|$.

(a) True

Γ ∵ $Q^T \cdot Q = I \Rightarrow Q^T = Q^T \cdot Q \cdot Q^{-1} = Q^{-1}$

∴ $Q \cdot Q^T = I \Rightarrow (Q^T)^T \cdot Q^T = I$

∴ $Q^{-1} = Q^T$ is an orthogonal matrix $\lrcorner$

(b) True

Γ ∵ $\|Qx\|^2 = (Qx)^T(Qx) = x^T Q^T Q x$

$= x^T x = \|x\|^2$

∴ $\|Qx\| = \|x\|$ $\lrcorner$