

## Quiz #4 (CSE 4190.313)

Monday, May 31, 2010

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1. (5 points) Suppose  $\mathbf{u}_1, \dots, \mathbf{u}_n$  and  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are orthonormal bases for  $\mathbb{R}^n$ . Construct the matrix  $A$  that transforms each  $\mathbf{v}_j$  into  $\mathbf{u}_j$  to give  $A\mathbf{u}_1 = \mathbf{v}_1, \dots, A\mathbf{u}_n = \mathbf{v}_n$ .

$$A = [\mathbf{v}_1 \ \dots \ \mathbf{v}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \mathbf{v}_1 \cdot \mathbf{u}_1^T + \dots + \mathbf{v}_n \cdot \mathbf{u}_n^T$$

2. (10 points) Explain why  $AA^+$  and  $A^+A$  are projection matrices. What fundamental subspaces do they project onto?

$$A = U \Sigma V^T, \quad A^+ = V \Sigma^+ U^T$$

$$\Rightarrow AA^+ = U \Sigma \Sigma^+ U^T = U \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$A^+A = V \Sigma^+ \Sigma V^T = V \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$\Rightarrow \textcircled{1} (AA^+)^2 = U \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= U \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} U^T = AA^+$$

$$\textcircled{2} (AA^+)^T = U \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} U^T = AA^+$$

Similarly,  $(A^+A)^2 = A^+A$ ,  $(A^+A)^T = A^+A$ .

$AA^+$  projects onto the column space of  $A$

$A^+A$  projects onto the row space of  $A$

3. (10 points) Give a quick reason why each of these statements is true:

- (a) (2 points) Every positive definite matrix is invertible.
- (b) (3 points) The only positive definite projection matrix is  $P = I$ .
- (c) (2 points) A diagonal matrix with positive diagonal entries is positive definite.
- (d) (3 points) A symmetric matrix with a positive determinant might not be positive definite!

(a) Eigenvalues  $\lambda_i > 0$

$$\det A = \lambda_1 \cdots \lambda_n > 0 \quad \therefore A \text{ is invertible}$$

(b)  $P^2 = P$  and  $P$  is invertible

$$\therefore P = P^{-1} P^2 = P^{-1} \cdot P = I$$

(c) For any  $x = (x_1, \dots, x_n)^T \neq 0$ ,

$$x^T D x = d_1 x_1^2 + \dots + d_n x_n^2 > 0.$$

(d) Counter Example:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} : \text{negative definite, but}$$

$$\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 > 0.$$

4. (5 points) For any symmetric matrix  $A$ , compute the ratio  $R(\mathbf{x})$  for the special choice  $\mathbf{x} = (1, \dots, 1)$ . How is the sum of all entries  $a_{ij}$  related to  $\lambda_1$  and  $\lambda_n$ ?

$$R(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \frac{\sum \sum a_{ij}}{n}$$

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n$$

$$\therefore n\lambda_1 \leq \sum \sum a_{ij} \leq n\lambda_n$$