Quiz #2 (CSE 4190.313)

Wednesday, April 4, 2012

Name: ______________________ **ID No:** ______________________

1. (4 points) Write all known relations between r and m and n if $A\mathbf{x} = \mathbf{b}$ has

   (a) (2 points) infinitely many solutions for every $\mathbf{b}$.

   (b) (2 points) exactly one solution for some $\mathbf{b}$, no solution for other $\mathbf{b}$.

(a) $m = r < n$ $\left(\because C(A) = \mathbb{R}^m, \text{ but } N(A) \neq \{0\}\right)$

(b) $m > n = r$ $\left(\because N(A) = \{0\}, \text{ but } C(A) \neq \mathbb{R}^m\right)$

2. (8 points) True or false (with a counterexample if false and a reason if true):

   (a) (2 points) A and A^T have the same left nullspace.

   (b) (2 points) If the row space equals the column space, then $A^T = A$.

   (c) (4 points) The solution $\mathbf{x}_p$ for $A\mathbf{x} = \mathbf{b}$ with all free variables zero is the shortest solution (minimum length $\|\mathbf{x}\|$).

(a) False $\left[\begin{array}{l} \because A = [1 \ 0], \quad A^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ N(A^T) = \{[0]\}, \ \dim N(A^T) = 0. \\ \text{But, } \dim\left(N\left((A^T)^T\right)\right) = 2 - 1 = 1 \neq 0 \end{array}\right]$

(b) False $\left[\begin{array}{l} \because A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow C(A^T) = C(A) = \mathbb{R}^2 \\ \text{But, } A^T \neq A \end{array}\right]$

(c) False

$$\left[\begin{array}{l} \because A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+u \\ u \end{bmatrix}, \quad u \in \mathbb{R} \\ \|\mathbf{x}\|^2 = (1+u)^2 + u^2 = 2\left(u + \frac{1}{2}\right)^2 + \frac{1}{2} \\ \min \|\mathbf{x}\| = \frac{1}{\sqrt{2}} \text{ when } u = -\frac{1}{2} \neq 0 \end{array}\right]$$

3. (3 points) Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(\mathbf{v})$ when $\mathbf{v} = (a,b)$.

$$\mathbf{v} = (a,b) = a(1,0) + b(0,1)$$

+2

$$= \tfrac{1}{2}a(2,0) + b\left[(1,1) - \tfrac{1}{2}(2,0)\right]$$

$$T(\mathbf{v}) = \tfrac{1}{2}a\,\cancel{T(2,0)} + b\,T(1,1) - \tfrac{1}{2}b\,\cancel{T(2,0)}$$

$$= b\,T(1,1) = (2b, 2b)$$

+1

4. (5 points) In the vector space P_3 of all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, let $\mathbf{S}$ be the subspace of polynomials with $\int_0^1 p(x)dx = 0$. Find a basis for the subspace $\mathbf{S}$.

$$\int_0^1 (a_0 + a_1x + a_2x^2 + a_3x^3)\,dx = 0$$

$$a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{3}a_2 + \tfrac{1}{4}a_3 = 0$$

+2

$a_1 = 1,\ a_2 = 0,\ a_3 = 0 \Rightarrow a_0 = -\frac{1}{2}$

$a_1 = 0,\ a_2 = 1,\ a_3 = 0 \Rightarrow a_0 = -\frac{1}{3}$

$a_1 = 0,\ a_2 = 0,\ a_3 = 1 \Rightarrow a_0 = -\frac{1}{4}$

+2

$\therefore\ x - \frac{1}{2},\ x^2 - \frac{1}{3},\ x^3 - \frac{1}{4}$ form a basis for the subspace S.

+1

5. (10 points) Suppose the matrices in $PA = LU$ are

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (1 point) What is the rank of A? 3 (= # of pivots)

(b) (1 point) What is a basis for the row space of A?

(c) (2 point) *True or false*: Rows 1, 2, 3 of A are linearly independent. False

(d) (1 point) What is a basis for the column space of A?

(e) (1 point) What is the dimension of the left nullspace of A? 1 (= 4−3)

(f) (4 point) What is the general solution to $A\mathbf{x} = \mathbf{0}$?

(c) False

$\because$ Rows 1, 2, 3 of PA are lin indep.

$\Rightarrow$ Rows 1, 2, 4 of A are lin. indep. ⌋

(f)

$$R = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 7 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ +2

Reduced Row Echelon Form

$$x = u \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u - 7v \\ u \\ 3v \\ v \\ 0 \end{bmatrix}$$

for all $u, v \in \mathbb{R}$

+2