Quiz #3 (CSE 4190.313)

Wednesday, April 18, 2012

Name: ______________________ **ID No:** ______________________

1. (5 points) Find a nonzero vector in both column spaces $C(A)$ and $C(B)$:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

$A\mathbf{x} = B\hat{\mathbf{x}}$ for some $\mathbf{x} = (x_1, x_2)^T$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$

$$\begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\ -x_2 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A\ B] \to U = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(-x_1, -x_2, \hat{x}_1\ \ \hat{x}_2)^T = (-3, -1, 1, 0)^T$ is a special solution

$$A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} \text{ is in } C(A) \cap C(B)$$

2. (6 points) Consider the problem of projecting a vector $\mathbf{b} = (b_1, b_2, b_3)^T$ onto the line through $\mathbf{a} = (1, 1, 1)^T$. We solve 3 equations $\mathbf{a}x = \mathbf{b}$ in 1 unknown (by least squares).

(a) (2 points) Solve $\mathbf{a}^T\mathbf{a}\hat{x} = \mathbf{a}^T\mathbf{b}$ to show that $\hat{x}$ is the average of b_i's.

(b) (2 point) Find $\mathbf{e} = \mathbf{b} - \mathbf{a}\hat{x}$, the variance $\|\mathbf{e}\|^2$, and the standard deviation $\|\mathbf{e}\|$.

(c) (2 points) The horizontal line $\hat{b} = 3$ is closest to $\mathbf{b} = (1, 2, 6)^T$. Check that $\mathbf{p} = (3, 3, 3)^T$ is perpendicular to $\mathbf{e}$ and find the projection matrix P.

(a) $\mathbf{a}^T\mathbf{a} = 3, \quad \mathbf{a}^T\mathbf{b} = b_1 + b_2 + b_3$

$$\Rightarrow \hat{x} = \frac{1}{3}(b_1 + b_2 + b_3) = \bar{b}$$

(b)

$$\mathbf{e} = \begin{bmatrix} b_1 - \bar{b} \\ b_2 - \bar{b} \\ b_3 - \bar{b} \end{bmatrix}$$

$$\|\mathbf{e}\|^2 = (b_1 - \bar{b})^2 + (b_2 - \bar{b})^2 + (b_3 - \bar{b})^2$$

$$\|\mathbf{e}\| = \sqrt{\sum_{i=1}^{3} (b_i - \bar{b})^2}$$

(c)

$$\mathbf{p} = \underbrace{\boxed{\frac{\mathbf{a}\cdot\mathbf{a}^T}{\mathbf{a}^T\cdot\mathbf{a}}}}_{\parallel \atop P}\mathbf{b} = \frac{\mathbf{a}}{3}(b_1 + b_2 + b_3) = 3\mathbf{a} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{p}^T \cdot \mathbf{e} = 3(b_1 - \bar{b}) + 3(b_2 - \bar{b}) + 3(b_3 - \bar{b})$$

$$= 3(b_1 + b_2 + b_3) - 9\bar{b} = 0$$

$$P = \frac{\mathbf{a}\cdot\mathbf{a}^T}{\mathbf{a}^T\cdot\mathbf{a}} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3. (3 points) By choosing the correct vector **b** in the Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \le n(a_1^2 + \cdots + a_n^2)$$

$\mathbf{a} = (a_1, \cdots, a_n)^T, \quad \mathbf{b} = (1, \cdots, 1)^T$

$\mathbf{a}^T \cdot \mathbf{b} = a_1 + \cdots + a_n, \quad \|\mathbf{a}\|^2 = a_1^2 + \cdots + a_n^2, \quad \|\mathbf{b}\|^2 = n$

$$|\mathbf{a}^T \cdot \mathbf{b}| \le \|\mathbf{a}\| \cdot \|\mathbf{b}\|$$

$$\Rightarrow (a_1 + \cdots + a_n)^2 \le n \cdot (a_1^2 + \cdots + a_n^2)$$

4. (6 points) Let $A = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$, and let V be the nullspace of A.

(a) (3 point) Find the projection matrix P_1 that projects vectors in R^3 onto $V^\perp$.

(b) (3 points) Find the projection matrix P_2 that projects vectors in R^3 onto V.

(a) $P_1 = A^T (AA^T)^{-1} A$: projection onto the row space of A

$$= \frac{1}{11} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 9 & 3 & -3 \\ 3 & 1 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

(b) $P_2 = I - P_1$

$$= \frac{1}{11} \begin{bmatrix} 2 & -3 & 3 \\ -3 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$