Quiz #4 (CSE 4190.313)

Monday, May 21, 2012

Name: ______________________ **ID No:** ______________________

1. (6 points) Suppose the only eigenvectors of A are multiples of $\mathbf{x} = (1, 0, 0)$. True or false, with a good reason or a counterexample.

   (a) (2 points) A is not invertible.

   (b) (2 points) A has a repeated eigenvalue.

   (c) (2 points) A is not diagonalizable.

(a) False

∵ Counterexample :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det A = 1 \neq 0$$

(b) True

∵ Otherwise, A has different eigenvalues and their corresponding eigenvectors are linearly independent. #

or

∵ $A\mathbf{x} = \lambda \mathbf{x} \Rightarrow A(c\mathbf{x}) = cA\mathbf{x} = \lambda(c\mathbf{x})$

∴ all multiples of $\mathbf{x}$ have the same eigenvalue.

(c) True

∵ Otherwise,

$$A = S\Lambda S^{-1} = S\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} S^{-1} = \lambda I$$

∴ A has e_1, e_2, e_3 as its eigenvectors #

2. (4 points) True or false, with a good reason or a counterexample. If the eigenvalues of A are 2, 2, 5, then the matrix is certainly

   (a) (2 points) diagonalizable.

   (b) (2 points) not diagonalizable.

(a) False

Counterexample :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

(b) False

Counterexample

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

3. (4 points)

   (a) (2 points) When do the eigenvectors for $\lambda = 0$ span the nullspace $N(A)$?

   (b) (2 points) When do all the eigenvectors for $\lambda \neq 0$ span the column space $C(A)$?

(a) There are always k $(= \dim N(A))$ independent eigenvectors for $\lambda = 0$, which span $N(A)$.

(b) When there are l $(= \dim C(A))$ independent eigenvectors for $\lambda \neq 0$, they span $C(A)$.

4. (6 points) True or false, with a good reason or a counterexample.

   (a) (3 points) A symmetric matrix cannot be similar to a nonsymmetric matrix.

   (b) (3 points) A cannot be similar to $-A$ unless $A = 0$.

(a) False

Γ∵ Counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}}_{= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}} = S \Lambda S^{-1}$$

(b) False

Γ∵ Counterexample:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}}$$

5. (5 points) Assuming A and B are square matrices, prove that AB has the same eigenvalues as BA.

For an eigenvalue λ of AB,

$$AB\mathbf{x} = \lambda \mathbf{x} \text{ for some } \mathbf{x} \neq \mathbf{0}.$$

$$\Rightarrow (BA)(B\mathbf{x}) = B(AB\mathbf{x}) = B(\lambda \mathbf{x}) = \lambda (B\mathbf{x})$$

① $B\mathbf{x} \neq \mathbf{0}$: λ is also an eigenvalue of BA.

② $B\mathbf{x} = \mathbf{0}$: $\lambda \mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \Rightarrow \lambda = 0$ ($\because \mathbf{x} \neq \mathbf{0}$)

B is singular ($\because B\mathbf{x} = \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$)

$\Rightarrow BA$ is also singular ($\because \det BA = 0$)

$\therefore$ 0 is also an eigenvalue of BA.