**Figure 2.6.** A device coordinate transformation makes Max a film star

**Exercise 2.26**

Suppose the window specified above is to be mapped onto a rectangular *device window* of the computer screen with lower left corner (200, 200) and upper right corner (600, 400). Determine the device coordinate transformation matrix.

### 2.7 Point and Line Geometry in Homogeneous Coordinates

The general equation of a line in the Cartesian plane is \( ax + by + c = 0 \). Suppose \((X, Y, W)\) are the homogeneous coordinates of the point \((x, y)\), so that \(x = X/W\) and \(y = Y/W\). Substituting for \(x\) and \(y\) in the equation of the line, and multiplying through by \(W\), yields the condition for \((X, Y, W)\) to be a point on the line

\[
aX + bY + cW = 0.
\]

(2.12)

The equation is known as the *homogeneous line equation*. The line is uniquely defined by the coefficients \(a, b,\) and \(c\), or any non-zero multiple \(ra, rb,\) and \(rc\) of them. Therefore, it is natural to specify the line by the homogeneous *line coordinates*

\[
\ell = (a, b, c).
\]

It is also useful to consider \(\ell\) to be a vector known as the *line vector*. Since any non-zero multiple of \(\ell\) defines the same line, only the direction of \(\ell\) is of importance. Let \(P(X, Y, W)\) be a point on the line. By permitting homogeneous coordinates \((X, Y, W)\) to be treated as a vector, equation (2.12) may be expressed as the dot product

\[
P \cdot \ell = aX + bY + cW = 0.
\]

(2.13)
The identity (2.13) leads to two useful operations: (i) determining the line through two distinct points, and (ii) determining the point of intersection of two lines.

To Find the Equation of the Line Through Two Points

Suppose \( \ell \) is the line vector of a line containing two distinct points \( \mathbf{P}_1(X_1, Y_1, Z_1) \) and \( \mathbf{P}_2(X_2, Y_2, Z_2) \). Then (2.13) yields

\[
\mathbf{P}_1 \cdot \ell = 0 \quad \text{and} \quad \mathbf{P}_2 \cdot \ell = 0.
\]

For any two vectors, the condition \( \mathbf{a} \cdot \mathbf{b} = 0 \) implies that \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular. Hence, \( \ell \) is a vector perpendicular to both \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \). To determine \( \ell \) it is sufficient to determine any vector perpendicular to \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \). In particular, the cross product gives a vector perpendicular to two given vectors, thus \( \ell = \mathbf{P}_1 \times \mathbf{P}_2 \) (or any multiple of \( \mathbf{P}_1 \times \mathbf{P}_2 \)). Hence, the equation of the line through two points can be determined by taking the ‘cross product’ of the homogeneous coordinates of the points.

Example 2.10

The line \( \ell \) passing through \((0, 5)\) and \((6, -7)\) satisfies

\[
\ell \cdot (0, 5, 1) = 0 \quad \text{and} \quad \ell \cdot (6, -7, 1) = 0.
\]

Hence

\[
\ell = (0, 5, 1) \times (6, -7, 1) = (12, 6, -30)
\]

giving the line \(12x + 6y - 30 = 0\).

To Determine the Point of Intersection of Two Lines

Suppose \( \mathbf{P} \) is the point of intersection of two lines \( \ell_1 \) and \( \ell_2 \). Then \( \mathbf{P} \) is a point on both lines and (2.13) yields

\[
\ell_1 \cdot \mathbf{P} = 0 \quad \text{and} \quad \ell_2 \cdot \mathbf{P} = 0.
\]

Hence \( \mathbf{P} \) is a vector perpendicular to both \( \ell_1 \) and \( \ell_2 \), and hence it is sufficient to take \( \mathbf{P} = \ell_1 \times \ell_2 \) (or any multiple of it). The cross product yields the homogeneous coordinates of the point of intersection.

Example 2.11

The point \( \mathbf{P} \) of intersection of the lines \( x - 7y + 8 = 0 \) and \( 3x - 4y + 1 = 0 \) satisfies

\[
(1, -7, 8) \cdot \mathbf{P} = 0 \quad \text{and} \quad (3, -4, 1) \cdot \mathbf{P} = 0.
\]
Hence
\[ \mathbf{P} = (1, -7, 8) \times (3, -4, 1) = (25, 23, 17). \]
The Cartesian coordinates of the intersection point are \((25/17, 23/17)\).

**Example 2.12**

The point \(\mathbf{P}\) of intersection of the lines \(2x - 5y = 0\) and \(2x - 5y + 3 = 0\) has homogeneous coordinates

\[ \mathbf{P} = (2, -5, 0) \times (2, -5, 3) = (-15, -6, 0). \]
The point of intersection \((-15, -6, 0)\) is a point at infinity since the lines are parallel.

**Exercises**

2.27 Determine the line passing through \((1, 3)\) and \((4, -2)\).

2.28 Determine the point of intersection of the lines \(x - 3y + 7 = 0\) and \(4x + 3y - 5 = 0\).

2.29 The methods used to determine the line through two distinct points and the point of intersection of two lines both involve the cross product. This is due to the duality between points and lines in the plane which relates results about points and lines to a dual result about lines and points. For example, the property 'points \(r_1, r_2,\) and \(r_3\) are collinear if and only if \(r_1 \cdot (r_2 \times r_3) = 0\)' has the dual property 'lines \(\ell_1, \ell_2,\) and \(\ell_3\) are concurrent if and only if \(\ell_1 \cdot (\ell_2 \times \ell_3) = 0\). Investigate further the property of duality [27, pp78-80].
4

Projections and the Viewing Pipeline

4.1 Introduction

This chapter describes the process of visualizing three-dimensional objects. Current display devices such as computer monitors and printers are two-dimensional, and therefore it is necessary to obtain a planar view of the object which gives the impression of the omitted third dimension. Visualization of an object is achieved by a sequence of operations called the viewing pipeline. Firstly, a projection is applied which maps the object to a new ‘flat’ object in a specified plane known as the viewplane. The ‘flat’ object represents a planar view of the object expressed in three-dimensional world coordinates. Secondly, a coordinate system in the viewplane is defined by specifying a point as origin, and two perpendicular vectors which give the directions of the coordinate axes. A viewplane coordinate mapping is applied to express the ‘flat’ object in terms of the chosen two-dimensional viewplane coordinate system. Finally, the ‘flat’ object is mapped to the computer screen by means of a two-dimensional device coordinate transformation.

![Diagram of the viewing pipeline]

Fig. 4.1. The viewing pipeline
The discussion begins in Section 4.2 with projections of the plane onto a line, and followed by projections of three-dimensional space onto a plane in Section 4.3. Section 4.4 introduces the viewpoint coordinate mapping which converts the three-dimensional world coordinate definition of the view to two-dimensional coordinates. The final step of mapping the view to the display device is discussed in Section 4.5.

4.2 Projections of the Plane

A view of a spatial object is obtained by a mapping or projection of three-dimensional space onto a plane. Consider first the simpler problem of projecting the plane onto a line contained in the plane. Let $\ell$ be a line in the plane, and let $V$ be a point not on the line. The perspective projection from $V$ onto $\ell$ is the transformation which maps any point $P$, distinct from $V$, onto the point $P'$ which is the intersection of the lines $\overline{VP}$ and $\ell$, as illustrated in Fig. 4.2. The point $V$ is called the viewpoint or centre of perspectivity, and the line $\ell$ is called the viewline. The next theorem shows that this mapping is indeed a transformation.

![Perspective projection](image)

**Fig. 4.2.** Perspective projection

**Theorem 4.1**

The perspective projection from the viewpoint $V$ (expressed in homogeneous coordinates) onto the viewline with line vector $\ell$ is the two-dimensional transformation given by the matrix $M = \ell^T V - (\ell \cdot V)I_3$, where $I_3$ denotes the $3 \times 3$ identity matrix.
Proof

Referring to Fig. 4.2, the image $P'$ of a point $P$ is obtained as the intersection of the viewline $\ell$ with the line through $V$ and $P$. The techniques of Section 2.7 imply that the line through $V$ and $P$ has the line vector $V \times P$, and therefore intersects $\ell$ in the point with homogeneous coordinates given by $\ell \times (V \times P)$. Applying the vector identity $A \times (B \times C) = (C \cdot A)B - (A \cdot B)C$, yields

$$P' = \ell \times (V \times P) = (P \cdot \ell)V - (\ell \cdot V)P.$$  

Replacing vectors by row matrices, and the dot product by a matrix multiplication, yields

$$P' = P\ell^T V - P(\ell \cdot V)I_3 = P(\ell^T V - (\ell \cdot V)I_3).$$

Thus $P' = PM$, where $M = \ell^T V - (\ell \cdot V)I_3$ as required.

□

Definition 4.1

The matrix $M$ is called the projection matrix of the perspective projection from $V$ onto $\ell$. Lines through the viewpoint are called projectors. The viewpoint $V$ can be a point at infinity in which case the projection is called a parallel projection. It is common practice to use the term ‘perspective projection’ to mean a non-parallel projection.

For a parallel projection with viewpoint $V(v_1,v_2,0)$ the projectors correspond to parallel lines in the Cartesian plane with direction $(v_1,v_2)$ as shown in Fig. 4.3.

![Fig. 4.3. Parallel projection in the direction $(v_1,v_2)$ onto the line $\ell$](image)

Example 4.1

The perspective projection of the triangle with vertices $A(2,3)$, $B(4,4)$, and $C(3,-1)$ onto the line $5x + y - 4 = 0$ from the viewpoint with Cartesian coordinates $(10,2)$ is illustrated in Fig. 4.4. The homogeneous viewpoint is
\( \mathbf{V} (10, 2, 1) \), the line vector is \( \ell = (5, 1, -4) \), and \( \ell \cdot \mathbf{V} = (5, 1, -4) \cdot (10, 2, 1) = 48 \). Hence

\[
\mathbf{M} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} (10 \ 2 \ 1) - 48 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 50 & 10 & 5 \\ 10 & 2 & 1 \\ -40 & -8 & -4 \end{pmatrix} - \begin{pmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 48 \end{pmatrix} = \begin{pmatrix} 2 & 10 & 5 \\ 10 & -48 & 1 \\ -40 & -8 & -52 \end{pmatrix}.
\]

The images of the vertices are obtained by multiplying the homogeneous coordinates of \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{C} \) by \( \mathbf{M} \). Then

\[
\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} \mathbf{M} = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 4 & 1 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} -6 & -126 & -39 \\ 8 & -152 & -28 \\ -44 & 68 & -38 \end{pmatrix}.
\]

The Cartesian coordinates of the vertex images are \( \mathbf{A}'(6/39, 126/39), \mathbf{B}'(-8/28, 152/28), \) and \( \mathbf{C}'(44/38, -68/38) \).  

![Perspective projection of Example 4.1](image.png)

**Fig. 4.4.** Perspective projection of Example 4.1

**Example 4.2**

The parallel projection of the triangle with vertices \( \mathbf{A}(2, 3), \mathbf{B}(4, 4), \) and \( \mathbf{C}(3, -1) \) onto the line \( 3x + 2y - 4 = 0 \) in the direction of the \( y \)-axis is shown in Fig. 4.5. The viewpoint is \( \mathbf{V} (0, 1, 0) \), the point at infinity in the direction of the \( y \)-axis. Then \( \ell = (3, 2, -4) \), and \( \ell \cdot \mathbf{V} = (3, 2, -4) \cdot (0, 1, 0) = 2 \). Thus

\[
\mathbf{M} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} (0 \ 1 \ 0) - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & -2 \end{pmatrix},
\]
and
\[
\begin{pmatrix}
A' \\
B' \\
C'
\end{pmatrix}
= \begin{pmatrix}
2 & 3 & 1 \\
4 & 4 & 1 \\
3 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
-2 & 3 & 0 \\
0 & 0 & 0 \\
0 & -4 & -2
\end{pmatrix}
M = \begin{pmatrix}
-4 & 2 & -2 \\
-8 & 8 & -2 \\
-6 & 5 & -2
\end{pmatrix}.
\]
Thus the Cartesian coordinates of the images are \( A'(2, -1), B'(4, -4), \) and \( C'(3, -5/2) \).

Fig. 4.5. Parallel projection of Example 4.2

Exercises

4.1 Determine the projection matrix for a perspective projection with viewpoint \((2, 11)\) and viewline \(-3x + 12y - 5 = 0\).

4.2 Determine the projection matrix for a parallel projection in the direction \((3, -2)\) and viewline \(7x - 5y - 2 = 0\).

4.3 Determine the projection matrix for a perspective projection with viewpoint \((7, -3)\) and viewline \(x - y + 9 = 0\). Apply the projection to the triangle with vertices \(A(2, 2), B(4, 3), \) and \(C(3, 5)\). Make a sketch showing the projection of the triangle onto the line.

4.4 Determine the projection matrix for a parallel projection in the direction \((-1, 4)\) and viewline \(2x - y + 8 = 0\). Apply the projection to the triangle with vertices \(A(2, 2), B(4, 3), \) and \(C(3, 5)\). Make a sketch showing the projection of the triangle onto the line.

4.5 Let \( P = (p_1, p_2, p_3), V = (v_1, v_2, v_3), \) and \( \ell = (a, b, c) \). Write out the proof of Theorem 4.1 in full.