

Chap 21. Numerics for ODEs

21.1 Methods for First-Order ODEs

Initial Value Problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

We compute approximate numeric values of $y(x)$ at

$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \quad \dots$$

$$\begin{aligned} y(x+h) &= y(x) + h y'(x) + \frac{1}{2} h^2 y''(x) + \dots \\ &\approx y(x) + h y'(x) = y(x) + h \cdot f(x, y) \end{aligned}$$

⇒

$$y_1 = y_0 + h f(x_0, y_0), \quad y_2 = y_1 + h f(x_1, y_1), \quad \dots$$

$$y_{n+1} = y_n + h f(x_n, y_n), \quad (n = 0, 1, 2, \dots)$$

↳ The Euler Method or the Euler-Cauchy Method

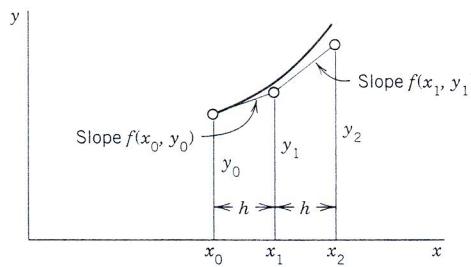


Fig. 448. Euler method

$$y(x+h) = y(x) + h y'(x) + \frac{1}{2} h^2 y''(x), \quad (x \leq x \leq x+h)$$

⇒ Local truncation error is $\Theta(h^2)$

⇒ Global error is proportional to $h^2(\frac{1}{h}) = h$
since the number of steps is proportional to $\frac{1}{h}$.

Improved Euler Method (Heun's Method)

$$y_{n+1}^* = y_n + h \cdot f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]$$

Local Error

The local error of the Improved Euler Method is $O(h^3)$

<proof>

$$\textcircled{a} \quad y(x_{n+h}) - y(x_n) = h \cdot \hat{f}_n + \frac{1}{2}h^2 \hat{f}'_n + \frac{1}{6}h^3 \hat{f}''_n + \dots$$

where $\hat{f}_n = f(x_n, y(x_n))$

$$\begin{aligned} \textcircled{b} \quad y_{n+1} - y_n &\approx \frac{1}{2}h \cdot [\hat{f}_n + \hat{f}_{n+1}] \\ &= \frac{1}{2}h \left[\hat{f}_n + (\hat{f}_n + h \cdot \hat{f}'_n + \frac{1}{2}h^2 \hat{f}''_n + \dots) \right] \\ &= h \cdot \hat{f}_n + \frac{1}{2}h^2 \hat{f}'_n + \frac{1}{4}h^3 \hat{f}''_n + \dots \end{aligned}$$

Hence, $\textcircled{b} - \textcircled{a}$ is $\frac{h^3}{12} \hat{f}'''_n + \dots$, which is $O(h^3)$

Ex 1 (Euler Method)

$$y' = f(x, y) = x + y, \quad y(0) = 0, \quad h = 0.2$$

$$\Rightarrow f(x_n, y_n) = x_n + y_n$$

$$y_{n+1} = y_n + 0.2(x_n + y_n)$$

Ex 3 (Improved Euler Method)

$$y' = f(x, y) = x + y, \quad y(0) = 0, \quad h = 0.2$$

$$\Rightarrow k_1 = 0.2(x_n + y_n)$$

$$k_2 = 0.2 f(x_{n+1}, y_{n+1}^*) = 0.2 (x_{n+1} + y_{n+1}^*)$$

$$= 0.2 (x_n + 0.2 + y_n + 0.2(x_n + y_n))$$

$$y_{n+1} = y_n + \frac{0.2}{2} (2.2x_n + 2.2y_n + 0.2)$$

$$= y_n + 0.22(x_n + y_n) + 0.02$$

Runge-Kutta Methods (RK Methods)

Classical Runge-Kutta Method of Fourth Order

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$k_3 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$\Rightarrow \begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

Ex 3 (Classical Runge-Kutta Method)

$$y' = f(x, y) = x + y, \quad h = 0.2$$

$$\Rightarrow k_1 = 0.2(x_n + y_n)$$

$$k_2 = 0.2(x_n + 0.1 + y_n + 0.5k_1) = 0.22(x_n + y_n) + 0.02$$

$$k_3 = 0.2(x_n + 0.1 + y_n + 0.5k_2) = 0.222(x_n + y_n) + 0.022$$

$$k_4 = 0.2(x_n + 0.2 + y_n + k_3) = 0.2444(x_n + y_n) + 0.0444$$

$$\Rightarrow y_{n+1} = y_n + 0.2214(x_n + y_n) + 0.0214$$

Fehlberg's fifth-order RK Method

$$y_{n+1} = y_n + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

Fehlberg's fourth-order RK Method

$$y_{n+1}^* = y_n + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

$$\text{where } k_1 = h f(x_n, y_n), \quad k_2 = h f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1)$$

$$k_3 = h f(x_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2),$$

21.3 Methods for Systems and Higher Order ODEs

Euler Method

$$y'' + 2y' + 0.75y = 0, \quad y(0) = 3, \quad y'(0) = -2.5, \quad h = 0.2$$

$$\Rightarrow y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$$\begin{cases} y_{1,n+1} = y_{1,n} + h f_1(x_n, y_{1,n}, y_{2,n}) \\ y_{2,n+1} = y_{2,n} + h f_2(x_n, y_{1,n}, y_{2,n}) \end{cases}$$

$$\begin{bmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -2y_2 - 0.75y_1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_{1,n+1} = y_{1,n} + 0.2 y_{2,n} \\ y_{2,n+1} = y_{2,n} + 0.2 (-2y_{2,n} - 0.75y_{1,n}) \end{cases}$$

See Table 21.11

The results are not accurate enough for practical purposes.

Runge-Kutta Methods

$$y(x_0) = y_0$$

$$\begin{cases} K_1 = h f(x_n, y_n) \\ K_2 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_1) \\ K_3 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_2) \\ K_4 = h f(x_n + h, y_n + K_3) \end{cases}$$

$$\Rightarrow y_{n+1} = y_n + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$