1. $x(t) = x_0 B_0^2(t) + x_1 B_1^2(t) + x_2 B_2^2(t)$

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$$= x_0(1-t)^2 + 2x_1(1-t)t + x_2 t^2$$

$$\rightarrow x^2(t) = x_0^2(1-t)^4 + 4x_1^2(1-t)^2t^2 + x_2^2t^4 + 4x_0x_1(1-t)^3t + 4x_1x_2(1-t)t^3 + 2x_0x_2(1-t)^2t^2$$

$$= x_0^2(1-t)^4 + \frac{1}{3}(2x_1^2 + x_0x_2)\cdot 6(1-t)^2t^2 + x_2^2t^4 + x_0x_1\cdot 4(1-t)^3t + x_1x_2\cdot 4(1-t)t^3$$

$$= x_0^2B_0^4(t) + x_0x_1B_1^4(t) + \frac{1}{3}(2x_1^2 + x_0x_2)B_2^4(t) + x_1x_2B_3^4(t) + x_2^2B_4^4(t)$$

Similarly,

$$y^2(t) = y_0^2B_0^4(t) + y_0y_1B_1^4(t) + \frac{1}{3}(2y_1^2 + y_0y_2)B_2^4(t) + y_1y_2B_3^4(t) + y_2^2B_4^4(t)$$

$$\therefore\ f(t) = \|\mathbf{x}(t)\|^2 = x^2(t) + y^2(t) = f_0B_0^4(t) + f_1B_1^4(t) + f_2B_2^4(t) + f_3B_3^4(t) + f_4B_4^4(t)$$

where

$$\begin{cases} f_0 = x_0^2 + y_0^2 & \text{2M, 1A} \\ f_1 = x_0x_1 + y_0y_1 & \text{2M, 1A} \\ f_2 = \frac{1}{3}\left[2(x_1^2+y_1^2) + (x_0x_2 + y_0y_2)\right] & \text{6M, 3A} \\ f_3 = x_1x_2 + y_1y_2 & \text{2M, 1A} \\ f_4 = x_2^2 + y_2^2 & \text{2M, 1A} \end{cases}$$

(multiplication) (addition)

Operation count : 14M, 7A

2. Given $f(t) = \|\mathbf{x}(t)\|^2 = \sum_{i=0}^{4} f_i B_i^4(t)$,

5

$$g(t) = \langle \mathbf{x}(t), \mathbf{x}'(t)\rangle = \frac{1}{2}f'(t) = \frac{1}{2}\sum_{i=0}^{3} 4(f_{i+1} - f_i)B_i^3(t) = \sum_{i=0}^{3} 2(f_{i+1} - f_i)B_i^3(t)$$

$$\Rightarrow g(t) = \sum_{i=0}^{3} g_i B_i^3(t) \quad \text{where } g_i = 2(f_{i+1} - f_i), \quad i = 0, 1, 2, 3$$

1M, 1A

Operation count : 4M, 4A

$\rightarrow$ Total operation count for the 'two-step' construction : 18M, 11A

(cont'd)

Alternatively, we can construct $g(t) = \langle \mathbf{x}(t), \mathbf{x}'(t) \rangle$ from scratch:

$$x(t) = x_0(1-t)^2 + 2x_1(1-t)t + x_2t^2$$

$$x'(t) = 2(x_1-x_0)(1-t) + 2(x_2-x_1)t$$

$$\rightarrow x(t)\,x'(t) = 2x_0(x_1-x_0)(1-t)^3 + 4x_1(x_1-x_0)(1-t)^2t + 2x_2(x_1-x_0)(1-t)t^2 + 2x_0(x_2-x_1)(1-t)^2t + 4x_1(x_2-x_1)(1-t)t^2 + 2x_2(x_2-x_1)t^3$$

$$= 2x_0(x_1-x_0)B_0^3(t) + \frac{2}{3}[2x_1(x_1-x_0) + x_0(x_2-x_1)]B_1^3(t) + \frac{2}{3}[x_2(x_1-x_0) + 2x_1(x_2-x_1)]B_2^3(t) + 2x_2(x_2-x_1)B_3^3(t)$$

Similar for $y(t)y'(t)$.

$$\therefore\ g(t) = \langle \mathbf{x}(t), \mathbf{x}'(t) \rangle = x(t)x'(t) + y(t)y'(t) = \sum_{i=0}^{3} g_i B_i^3(t)$$

where

$$\begin{cases} g_0 = 2[x_0(x_1-x_0) + y_0(y_1-y_0)] & 3M,\ 1A \\ g_1 = \frac{2}{3}[2\{x_1(x_1-x_0) + y_1(y_1-y_0)\} + \{x_0(x_2-x_1) + y_0(y_2-y_1)\}] & 6M,\ 3A \\ g_2 = \frac{2}{3}[\{x_2(x_1-x_0) + y_2(y_1-y_0)\} + 2\{x_1(x_2-x_1) + y_1(y_2-y_1)\}] & 6M,\ 3A \\ g_3 = 2[x_2(x_2-x_1) + y_2(y_2-y_1)] & 3M,\ 1A \end{cases}$$

Common subexpressions: (x_1-x_0), (x_2-x_1), (y_1-y_0), (y_2-y_1) $\rightarrow$ 4A

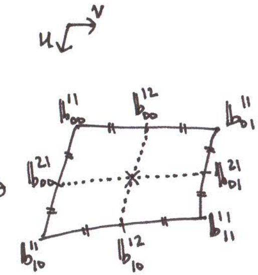
$\rightarrow$ Operation count: 18M, 12A

Thus, the construction of $g(t)$ from $f(t)$ takes one less addition operation than a direct one.

(18M, 11A) (18M, 12A)

3. Idea)

$$\begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} & \mathbf{b}_{02} \\ \mathbf{b}_{10} & \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{20} & \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \xrightarrow[\text{in } u\text{-dir.}]{\text{de Casteljau}} \begin{bmatrix} \mathbf{b}^{10}_{00} & \mathbf{b}^{10}_{01} & \mathbf{b}^{10}_{02} \\ \mathbf{b}^{10}_{10} & \mathbf{b}^{10}_{11} & \mathbf{b}^{10}_{12} \end{bmatrix} \xrightarrow[\text{in } v\text{-dir.}]{\text{de Casteljau}} \begin{bmatrix} \mathbf{b}^{11}_{00} & \mathbf{b}^{11}_{01} \\ \mathbf{b}^{11}_{10} & \mathbf{b}^{11}_{11} \end{bmatrix} \xrightarrow[\text{patch}]{\text{bilinear}}$$



sol)

$$[\mathbf{b}^{10}_{ij}]_{2\times3} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} & \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 3/2 \\ 3/2 \end{bmatrix} & \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix} & \begin{bmatrix} 2 \\ 3/2 \\ 3/2 \end{bmatrix} \end{bmatrix}, \quad [\mathbf{b}^{11}_{ij}]_{2\times2} = \begin{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} & \begin{bmatrix} 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ \begin{bmatrix} 1/2 \\ 3/2 \\ 3/2 \end{bmatrix} & \begin{bmatrix} 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} \end{bmatrix}, \quad \mathbf{b}^{21}_{00} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{b}^{12}_{00} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{b}^{21}_{01} = \begin{bmatrix} 3/2 \\ 1 \\ 1 \end{bmatrix}. \quad \mathbf{b}^{12}_{10} = \begin{bmatrix} 1 \\ 3/2 \\ 3/2 \end{bmatrix},$$

At $(u, v) = (0.5, 0.5)$,

biquadratic patch

$$\begin{cases} \mathbf{x}_u = 2\Delta \mathbf{b}^{12}_{00} = 2(\mathbf{b}^{12}_{10} - \mathbf{b}^{12}_{00}) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \\ \mathbf{x}_v = 2\Delta \mathbf{b}^{21}_{00} = 2(\mathbf{b}^{21}_{01} - \mathbf{b}^{21}_{00}) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{n} = \dfrac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \end{cases}$$