Offsets, Sweeps, and Minkowski sums for Freeform Curves and Surfaces

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Motivation

Dialation, Erosion, and Medial Axis
Motivation

Tool Path Generation for NC Machining

\[ C_r(t) = C(t) + r \cdot N(t), \]
\[ N(t) = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}. \]
Motivation

Collision Avoidance Motion Planning

\[ A \cap (B + p) \neq \emptyset \]
\[ a = b + p \]
\[ p = a - b \]
\[ p \in A - B \]
Motivation

Minimum Distance Computation

\[ A - B = \{ a - b \mid a \in A, b \in B \} \]
Motivation

Natural Shape Design and Compact Shape Representation

In these special cases, the sweep surfaces are rational
Outline

Introduction
Research Issues
Non-rational Envelope of Curves and Surfaces
Rational Envelope of Lines and Planes
Offset Trimming
Conclusions
Introduction

• Conventional Research in CAGD
  Design and representation of freeform geometry

• Geometric Operations
  Offsets, Minkowski sums, sweeps
  Medial axis transformation, bisectors
  Voronoi diagrams and Voronoi cells

• Fundamental Difficulties
  Results are often non-rational curves/surfaces
  Arrangement of algebraic varieties
  High degree, robustness, efficiency, etc
Definitions

Offset of $A$

$$A \uparrow r = \bigcup_{p \in A} O_r(p)$$

Minkowski Sum of $A$ and $B$

$$A \oplus B = \{a + b \mid a \in A, b \in B\}$$
$$= \bigcup_{a \in A} (B + a)$$
$$= \bigcup_{b \in B} (A + b)$$

$$A \uparrow r = A \oplus O_r(0)$$
Envelope Curve

\[ C_1(u) = (x_1(u), y_1(u)) : \text{Trajectory} \]
\[ C_2(v) = (x_2(v), y_2(v)) : \text{Moving curve} \]
\[ (x(u, v), y(u, v)) : \text{Envelope curve defined by} \]

\[
x(u, v) = x_1(u) + x_2(v),
\]
\[
y(u, v) = y_1(u) + y_2(v),
\]
\[
F(u, v) = x'_1(u)y'_2(v) - y'_1(u)x'_2(v) = 0.
\]
Envelope Curve Equation

\[
x = x_1(u) + x_2(v) \\
y = y_1(u) + y_2(v) \\
0 = x'_1(u)y'_2(v) - y'_1(u)x'_2(v)
\]

Eliminating \( u \) and \( v \), the envelope curve \( e(x, y) = 0 \) has algebraic degree \( O(d^3) \) much higher than \( (2d - 2) \) of \( F(u, v) = 0 \).
\[ \left\langle (x, y) - C_1(u), C'_1(u) \right\rangle = 0, \]
\[ \left\langle (x, y) - C_2(v), C'_2(v) \right\rangle = 0, \]
\[ \left\langle (x, y) - \frac{C_1(u) + C_2(v)}{2}, C_1(u) - C_2(v) \right\rangle = 0. \]
Bisector Equation

\[
\left\langle (x, y) - C_1(u), C'_1(u) \right\rangle = 0, \\
\left\langle (x, y) - C_2(v), C'_2(v) \right\rangle = 0, \\
\left\langle (x, y) - \frac{C_1(u) + C_2(v)}{2}, C_1(u) - C_2(v) \right\rangle = 0.
\]

Eliminating \( u \) and \( v \), the curve \( b(x, y) = 0 \) has degree \( 7d_1d_2 - 3(d_1 + d_2) + 1 \)

Eliminating \( x \) and \( y \), we have \( F(u, v) = 0 \) of degree \( 2(d_1 + d_2) - 2 \).

For \( d_1 = d_2 = 3 \), \( F(u, v) = 0 \) has degree 10 and \( b(x, y) = 0 \) has degree 46
Sweep Envelope Surface

\[ S(u, v) = (s_1(u, v), s_2(u, v), s_3(u, v))^T \]

\[ T(u, v, t) \]

\[
= (x(u, v, t), y(u, v, t), z(u, v, t))^T
\]

\[
= \begin{bmatrix}
a_{11}(t) & a_{12}(t) & a_{13}(t) \\
a_{21}(t) & a_{22}(t) & a_{23}(t) \\
a_{31}(t) & a_{32}(t) & a_{33}(t)
\end{bmatrix}
\begin{bmatrix}
s_1(u, v) \\
s_2(u, v) \\
s_3(u, v)
\end{bmatrix}
+ \begin{bmatrix}
c_1(t) \\
c_2(t) \\
c_3(t)
\end{bmatrix}
\]

\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t}
\end{vmatrix}
= 0
\]
Calculus on Envelope Curve

\[ F(u, v) = 0 \]
\[ F_u + F_v \frac{dv}{du} = 0 \]
\[ F_{uu} + 2F_{uv} \frac{dv}{du} + F_{vv} \left( \frac{dv}{du} \right)^2 + F_v \frac{d^2v}{du^2} = 0 \]

\[ \frac{dv}{du} = -\frac{F_u}{F_v} \]
\[ \frac{d^2v}{du^2} = -\frac{F_{uu} + 2F_{uv} \frac{dv}{du} + F_{vv} \left( \frac{dv}{du} \right)^2}{F_v} \]
\[ = -\frac{F_{uu}F_v^2 + 2F_{uv}F_uF_v - F_{vv}F_u^2}{F_v^3} \]

\[ x = x_1(u) + x_2(v), \]
\[ y = y_1(u) + y_2(v), \]
\[ F(u, v) = x'_1(u)y'_2(v) - y'_1(u)x'_2(v) = 0. \]
Calculus on Envelope Curve

\[
\frac{dx}{du} = x_u(u, v) + x_v(u, v) \frac{dv}{du}
\]
\[
\frac{dy}{du} = y_u(u, v) + y_v(u, v) \frac{dv}{du}
\]
\[
\frac{d^2x}{du^2} = x_{uu} + 2x_{uv} \frac{dv}{du} + x_{vv} \left( \frac{dv}{du} \right)^2 + x_v \frac{d^2v}{du^2}
\]
\[
\frac{d^2y}{du^2} = y_{uu} + 2y_{uv} \frac{dv}{du} + y_{vv} \left( \frac{dv}{du} \right)^2 + y_v \frac{d^2v}{du^2}
\]

\[\kappa(u) = \frac{\frac{dx}{du} \frac{d^2y}{du^2} - \frac{d^2x}{du^2} \frac{dy}{du}}{\left[ \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 \right]^{3/2}}\]

\[x = x_1(u) + x_2(v),\]
\[y = y_1(u) + y_2(v),\]
\[F(u, v) = x_1'(u)y_2'(v) - y_1'(u)x_2'(v) = 0.\]
Envelope Surface

Given two surfaces
\[ S_1(u, v) = (x_1(u, v), y_1(u, v), z_1(u, v)) \]
\[ S_2(s, t) = (x_2(s, t), y_2(s, t), z_2(s, t)) \]

Envelope surface \((x(u, v, s, t), y(u, v, s, t), z(u, v, s, t))\) is defined by

\[
x(u, v, s, t) = x_1(u, v) + x_2(s, t)
\]
\[
y(u, v, s, t) = y_1(u, v) + y_2(s, t)
\]
\[
z(u, v, s, t) = z_1(u, v) + z_2(s, t)
\]

\[ F(u, v, s, t) = \left\langle N_1(u, v), \frac{\partial S_2}{\partial s}(s, t) \right\rangle = 0 \]

\[ G(u, v, s, t) = \left\langle N_1(u, v), \frac{\partial S_2}{\partial t}(s, t) \right\rangle = 0 \]
Rational Envelope of Line Sweep

\[\begin{align*}
a(t)x + b(t)y + c(t) &= 0, \\
a'(t)x + b'(t)y + c'(t) &= 0.
\end{align*}\]

\[\begin{align*}
x &= \frac{b(t)c'(t) - b'(t)c(t)}{a(t)b'(t) - a'(t)b(t)}, \\
y &= \frac{a'(t)c(t) - a(t)c'(t)}{a(t)b'(t) - a'(t)b(t)}.
\end{align*}\]
Rational Envelope of Plane Sweep

One-parameter family of planes produces a rational developable surface

\[ a(t)x + b(t)y + c(t)z + d(t) = 0, \]
\[ a'(t)x + b'(t)y + c'(t)z + d'(t) = 0. \]

Two-parameter family of planes produces a rational envelope surface

\[ a(u, v)x + b(u, v)y + c(u, v)z + d(u, v) = 0, \]
\[ a_u(u, v)x + b_u(u, v)y + c_u(u, v)z + d_u(u, v) = 0, \]
\[ a_v(u, v)x + b_v(u, v)y + c_v(u, v)z + d_v(u, v) = 0. \]
Trimming Offset Curve
Trimming Offset Surface
Trimming Offset Surface
Conclusions

• Problem reduction to a system of equations in the parameter space
• Degree reduction in the parameter space
• Dimension reduction to the parameter space
• Squared-distance-based formulation for offset trimming