Engineering Mathematics I
(Comp 400.001)

Midterm Exam II: May 16, 2002

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Name: ____________________

ID No: ____________________

Dept: ____________________

E-mail: ____________________

1. (5 points) Find a good way to compute

$$\sqrt{x^2+16}-4$$

for small $|x|$.

$$\frac{x^2}{\sqrt{x^2+16}+4}$$ +5

2. (10 points) Show that two similar $n \times n$ matrices A and B have the same eigenvalues. (A and B are called *similar* if and only if there is a nonsingular $n \times n$ matrix T such that $B = T^{-1}AT$.)

$B = T^{-1}AT \Rightarrow A = TBT^{-1}$ +2

$Ax = \lambda x$, for some $x \neq 0$ +2

$TBT^{-1}x = \lambda x$ +2

$BT^{-1}x = T^{-1}(\lambda x) = \lambda(T^{-1}x)$ +2

$By = \lambda y$, where $\underline{y = T^{-1}x \neq 0}$ +2

$\Gamma \because$ If $y = T^{-1}x = 0$

$x = TT^{-1}x = 0$ #$\lrcorner$

$\therefore \lambda$ is an eigenvalue of B.

3. (10 points) Compute $\sinh 0.3$ from $\sinh(-0.5) = -0.521$, $\sinh 0 = 0$, and $\sinh 1 = 1.175$ by quadratic interpolation

$$(x_0, f_0) = (-0.5, -0.521)$$

$$(x_1, f_1) = (0, 0)$$

$$(x_2, f_2) = (1, 1.175)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-0.5)(-1.5)} = \frac{4}{3}x(x-1)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+0.5)(x-1)}{(0.5)(-1)} = -2(x+0.5)(x-1)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+0.5)x}{(1.5)(1)} = \frac{2}{3}x(x+0.5)$$

$$\sinh 0.3 \approx P_2(0.3)$$

$$= -0.521 * L_0(0.3) + 0 * L_1(0.3) + 1.175 * L_2(0.3)$$

$$= -0.521 * \frac{4}{3}(0.3)(-0.7) + 1.175 \cdot \frac{2}{3}(0.3)(0.8)$$

$$= (-0.521) * (-0.28) + (1.175) * (0.16)$$

$$= 0.33388$$

4. (20 points) Fit a cubic parabola $p(x) = a + bx + cx^2 + dx^3$ by least squares to

$$(-2,-6),\ (-1,-2),\ (0,-1),\ (1,0),\ (2,10),\ (4,78).$$

Set up a matrix equation; and you don't have to solve the matrix equation itself. Show your work for partial credit.

$$g(a,b,c,d) = \sum_{i=1}^{6} (y_i - a - bx_i - cx_i^2 - dx_i^3)^2$$ +5

$$\frac{\partial g}{\partial a} = -2\sum (y_i - a - bx_i - cx_i^2 - dx_i^3) = 0$$

$$\frac{\partial g}{\partial b} = -2\sum x_i (y_i - a - bx_i - cx_i^2 - dx_i^3) = 0$$ +5

$$\frac{\partial g}{\partial c} = -2\sum x_i^2 (y_i - a - bx_i - cx_i^2 - dx_i^3) = 0$$

$$\frac{\partial g}{\partial d} = -2\sum x_i^3 (y_i - a - bx_i - cx_i^2 - dx_i^3) = 0$$

$$an + b\sum x_i + c\sum x_i^2 + d\sum x_i^3 = \sum y_i$$
$$a\sum x_i + b\sum x_i^2 + c\sum x_i^3 + d\sum x_i^4 = \sum x_i y_i$$ +5
$$a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 + d\sum x_i^5 = \sum x_i^2 y_i$$
$$a\sum x_i^3 + b\sum x_i^4 + c\sum x_i^5 + d\sum x_i^6 = \sum x_i^3 y_i$$

$$\begin{bmatrix} 6 & 4 & 26 & 64 \\ 4 & 26 & 64 & 290 \\ 26 & 64 & 290 & 1024 \\ 64 & 290 & 1024 & 4226 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 79 \\ 346 \\ 1262 \\ 5122 \end{bmatrix}$$ +5

5. (15 points) Table 1 shows the result of applying the Improved Euler method to the following initial value problem with $h = 0.25$:

$$y' = 1 + y/x, \quad \text{for } 1 \le x \le 2, \quad y(1) = 2.$$

Fill in the blank; and show your work for partial credit.

x_i	y_i
1.25	2.7750000
1.50	3.60083
1.75	4.4688294
2.00	5.3728586

Table 1: Improved Euler Method

$x_1 = 1.25, \quad y_1 = 2.775$

$h = 0.25, \quad f(x,y) = 1 + \frac{y}{x}$ +2

$k_1 = 0.25 * f(x_1, y_1)$

$= 0.25 * \left(1 + \frac{2.775}{1.25}\right)$

$= 0.805$ +5

$k_2 = 0.25 * f(x_2, y_1 + k_1)$

$= 0.25 * \left(1 + \frac{2.775 + 0.805}{1.5}\right)$

$= 0.846667$ +5

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$= 2.775 + \frac{1}{2}(0.805 + 0.846667)$

$= 3.60083$ +3

6. (20 points) Table 2 shows the result of applying the Runge-Kutta method to the following initial value problem with $h = 0.2$:

$$y' = y - x^2 + 1, \quad \text{for } 0 \le x \le 1, \quad y(0) = 0.5.$$

Fill in the blank; and show your work for partial credit.

x_i	y_i
0.2	0.8292933
0.4	1.21408
0.6	1.6489220
0.8	2.1272027
1.0	2.6408227

Table 2: Runge-Kutta Method

$x_1 = 0.2, \quad y_1 = 0.8292933$
$h = 0.2, \quad f(x,y) = 1 - x^2 + y$ +2

$k_1 = 0.2 * f(x_1, y_1) = 0.2 * (1 - 0.2^2 + 0.8292933)$ +4
$= 0.357859$

$k_2 = 0.2 * f(x_1 + 0.1, y_1 + 0.5k_1)$
$= 0.2 * (1 - 0.3^2 + 0.8292933 + 0.178930)$ +4
$= 0.383645$

$k_3 = 0.2 * f(0.3, y_1 + 0.5k_2)$
$= 0.2 * (1 - 0.3^2 + 0.8292933 + 0.191823)$ +4
$= 0.386223$

$k_4 = 0.2 * f(x_2, y_1 + k_3)$
$= 0.2 * (1 - 0.4^2 + 0.8292933 + 0.386223)$ +4
$= 0.411103$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.21408$ +2

7. (20 points) Using $h = 0.5$ and $k = 0.5$, approximate the solution to the following elliptic equation (**warning: this is different from** $u_{xx} + u_{yy} = 4$**!**)

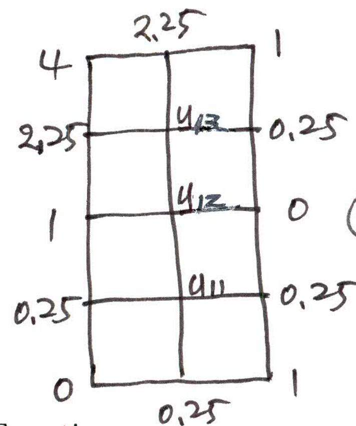
$$u_{xx} + 2u_{yy} = 4, \quad 0 < x < 1,\ 0 < y < 2$$

with boundary conditions:

$$u(x,0) = x^2, \quad u(x,2) = (x-2)^2, \quad 0 \le x \le 1;$$
$$u(0,y) = y^2, \quad u(1,y) = (y-1)^2, \quad 0 \le y \le 2.$$

Set up a matrix equation. (You don't have to solve the matrix equation itself.) Show your work for partial credit.

i	j	x_i	y_j	$u(x_i, y_j)$
1	1	0.5	0.5	
1	2	0.5	1.0	
1	3	0.5	1.5	



+5

Table 3: Approximate Solution to Poisson Equation

$$\frac{u(E) - 2u(x,y) + u(W)}{h^2} + 2\cdot\frac{u(N) - 2u(x,y) + u(S)}{k^2} = 4$$

$$u(E) + u(W) + 2u(N) + 2u(S) - 6u(x,y) = 0.25 * 4 = 1$$ +7

$j=1$: $-6u_{11} + 2u_{12} = 1 - 1 = 0$

$j=2$: $2u_{11} - 6u_{12} + 2u_{13} = 1 - 1 = 0$

$j=3$: $2u_{12} - 6u_{13} = 1 - 7 = -6$

+8

$$\begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$