11.7 Fourier Integral

Ex1 (Square Wave)

$$f_L(x) = \begin{cases} 0 & \text{if } -L<x<-1, \\ 1 & \text{if } -1<x<1, \\ 0 & \text{if } 1<x<L, \end{cases} \qquad f_L(x+2L) = f_L(x)$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Since f_L is even, $b_n = 0$ for all n

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^1 \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}$$

$\hookrightarrow$ amplitude spectrum of f_L

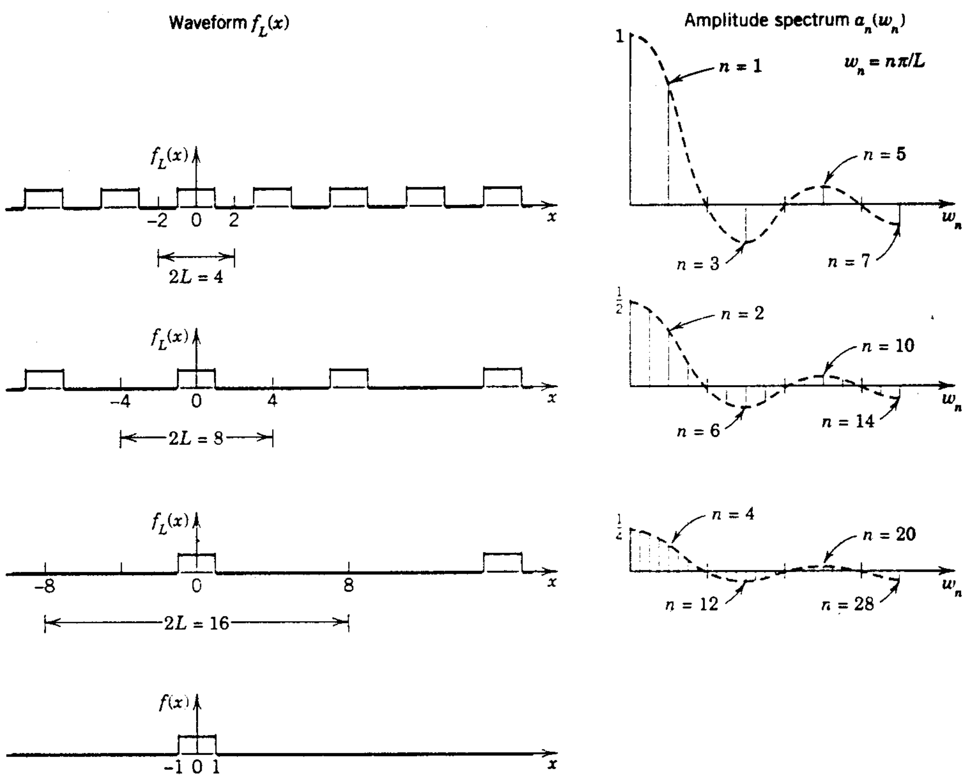


Fig. 254. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv$$
$$+ \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv$$
$$+ \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]$$

Let $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$\Rightarrow \quad \frac{1}{L} = \frac{\Delta w}{\pi}$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv$$
$$+ (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]$$

Assuming

$f(x) = \lim_{L\to\infty} f_L(x)$: absolutely integrable $\left(\text{i.e., } \int_{-\infty}^{\infty} |f(x)|\,dx < \infty\right)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv$$
$$+ \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\Big]\,dw$$

$$= \int_0^{\infty}\big[A(w)\cos wx + B(w)\sin wx\big]\,dw \; : \text{Fourier Integral}$$

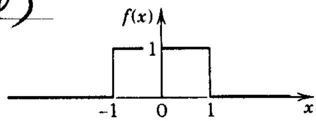
where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

Applications

Ex2 (Single Pulse, Sine Integral)

$$f(x)=\begin{cases}1 & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$$



$$\Rightarrow A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv=\frac{2\sin w}{\pi w}$$

$$B(w)=\frac{1}{\pi}\int_{-1}^{1} 1\cdot\sin wv\,dv=0$$

$$\Rightarrow f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw$$

$$\Rightarrow \int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw=\begin{cases}\pi/2 & \text{if } 0\le x<1\\ \pi/4 & \text{if } x=1\\ 0 & \text{if } x>1\end{cases}$$

$$\left[\frac{1}{2}\big(f(1-0)+f(1+0)\big)=\frac{1}{2}\right]$$

If $x=0$, $\displaystyle\int_0^{\infty}\frac{\sin w}{w}\,dw=\frac{\pi}{2}$.

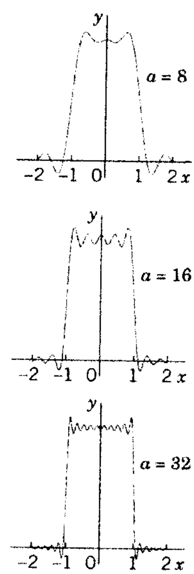
In general,

$$\int_0^{a}\frac{\cos wx\sin w}{w}\,dw$$

approximates

$$\int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw$$

as $a\to\infty$



Fourier Cosine and Sine Integrals

If $f(x)$ is even, $B(w)=0$

$$A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$$

$$f(x)=\int_0^\infty A(w)\cos wx\,dw$$

If $f(x)$ is odd, $A(w)=0$

$$B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$$

$$f(x)=\int_0^\infty B(w)\sin wx\,dw$$

11.8 Fourier Cosine and Sine Transforms

Fourier Cosine Transforms

If $f(x)$ is even,

$$f(x)=\int_0^\infty A(w)\cos wx\,dw, \quad A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$$

Let $\hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$: Fourier cosine Transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw$$

(Notation: $\mathcal{F}_c(f)=\hat{f}_c$)

Fourier Sine Transforms

If $f(x)$ is odd,

$$f(x)=\int_0^\infty B(w)\sin wx\,dw, \quad B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$$

Let $\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$: Fourier sine Transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\,dw$$

(Notation: $\mathcal{F}_s(f)=\hat{f}_s$)

Ex1: $f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

$\Rightarrow \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$

$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos aw}{w}\right)$

Linearity, Transforms of Derivatives

(a) $\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$

(b) $\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$

Th1: $f(x)$: conti. and absolutely integrable on the x-axis

$f'(x)$: piecewise conti on each finite interval

$f(x) \to 0$ as $x \to \infty$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$,

ⓑ $\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

ⓑ $\mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, w f(0)$

Ex3

$f(x) = e^{-ax}$, where $a > 0$

$\Rightarrow (e^{-ax})'' = a^2 e^{-ax}$

$f''(x) = a^2 f(x)$

$\Rightarrow a^2\mathcal{F}_c\{f(x)\} = \mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

$(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{\frac{2}{\pi}}$

$\therefore \mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right)$, $a > 0$

11.9 Fourier Transform

Complex Form of the Fourier Integral

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

where
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$$
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv}_{\text{even function of } w}\Big] dw$$

Note that
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv}_{\text{odd function of } w}\Big] dw = 0$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)[\cos(wx - wv) + i\sin(wx - wv)]\, dv\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cdot e^{iw(x-v)}\, dv\, dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\cdot e^{-iwv}\, dv\right] e^{iwx}\, dw$$

$\Rightarrow$ Let

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-iwx}\, dx \quad : \text{ Fourier Transform}$$

Then
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw \quad : \text{ Inverse Fourier Transform}$$

Linearity, Fourier Transform of Derivatives

Th1

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Th2: $f(x)$: conti on the x-axis

$f(x) \to 0$ as $|x| \to \infty$

$f'(x)$: abs. integrable on the x-axis

$\Rightarrow$ ∘. $\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$

∘. $\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}$

Ex3:

$$\mathcal{F}(xe^{-x^2}) = \mathcal{F}\left\{-\frac{1}{2}(e^{-x^2})'\right\} = -\frac{1}{2}\cdot iw\mathcal{F}(e^{-x^2})$$

$$= -\frac{1}{2}iw\cdot\frac{1}{\sqrt{2}}e^{-w^2/4} = -\frac{iw}{2\sqrt{2}}e^{-w^2/4}$$

Convolution

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

Th4

$f(x), g(x)$: piecewise conti, bounded,

absolutely integrable

$$\Rightarrow \mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$$

$$\therefore f*g = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}dw$$