

Engineering Mathematics I

(Comp 400.001)

Final Exam: June 19, 2001

1. (15 points) Show that the given integral represents the indicated function.

$$\int_0^{\infty} \left[\left(\frac{\sin 2w}{w} \right) \cos wx + \left(\frac{1 - \cos 2w}{w} \right) \sin wx \right] dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi & \text{if } 0 < x < 2 \\ \pi/2 & \text{if } x = 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

$$\begin{aligned} \Rightarrow A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos wx \, dx = \frac{1}{\pi} \int_0^2 \pi \cdot \cos wx \, dx \\ &= \left[+ \frac{1}{w} \sin wx \right]_0^2 = \frac{\sin 2w}{w} \end{aligned}$$

$$\begin{aligned} B(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin wx \, dx = \int_0^{\pi} \sin wx \, dx \\ &= \left[-\frac{1}{w} \cos wx \right]_0^{\pi} = \frac{1 - \cos 2w}{w} \end{aligned}$$

2. (15 points)

(a) Find the Fourier series of the following periodic function:

$$f(x + 2\pi) = f(x) = x^2, \quad \text{for } -\pi < x < \pi$$

(b) Using this result, show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}$$

(a) $f(x)$: even function

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{1}{3} \pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[\frac{1}{n} x^2 \sin nx \Big|_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin nx dx \right]$$

$$= -\frac{4}{n\pi} \left[-\frac{1}{n} x \cos nx \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx dx \right]$$

$$= \frac{4\pi}{n^2\pi} \cos n\pi - \frac{4}{n^2\pi} \left[\frac{1}{n} \sin nx \right]_0^{\pi}$$

$$= \frac{4}{n^2} (-1)^n$$

$$f(x) = \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cdot \cos nx$$

$$(b) \quad 0 = f(0) = \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^{n+1} = \frac{1}{3} \pi^2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{12} \pi^2$$

Quiz #7 (CSE 400.001)

Thursday, June 7, 2001

Name: _____ E-mail: _____

Dept: _____ ID No: _____

1. (7 points) Compute the Fourier series of the following periodic function, of period $p = 2L = 2$:

$$f(x) = x + x^2, \quad -1 < x < 1.$$

$$f(x) = f_1(x) + f_2(x), \quad \text{where } f_1(x) = x, \quad f_2(x) = x^2$$

$$f_1(x) = x: \text{ odd} \Rightarrow b_n = 2 \int_0^1 x \sin n\pi x dx = -\frac{2}{n\pi} \cos n\pi$$

$$\therefore f_1(x) = \frac{2}{\pi} \left(\sin \pi x - \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x - \dots \right)$$

(+2)

$$f_2(x) = x^2: \text{ even} \Rightarrow a_0 = \int_0^1 x^2 dx = \frac{1}{3}$$

$$a_n = 2 \int_0^1 x^2 \cos n\pi x dx = \frac{4}{(n\pi)^2} \cos n\pi$$

$$\therefore f_2(x) = \frac{1}{3} + \frac{4}{\pi^2} \left(-\cos \pi x + \frac{1}{2} \cos 2\pi x - \frac{1}{3} \cos 3\pi x + \dots \right)$$

(+2)

$$f(x) = f_1(x) + f_2(x)$$

$$= \frac{1}{3} + \frac{2}{\pi} \left(\sin \pi x - \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x - \dots \right)$$

$$+ \frac{4}{\pi^2} \left(-\cos \pi x + \frac{1}{2} \cos 2\pi x - \frac{1}{3} \cos 3\pi x + \dots \right)$$

(+3)

2. (8 points) Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} x+x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (x+x^2) e^{-i\omega x} dx \quad (+1)$$

$$\int_{-1}^1 (x+x^2) e^{-i\omega x} dx = \frac{1}{-i\omega} (x+x^2) e^{-i\omega x} \Big|_{-1}^1 + \frac{1}{i\omega} \int_{-1}^1 (1+2x) e^{-i\omega x} dx \quad (+2)$$

$$= \frac{1}{-i\omega} 2e^{-i\omega} + \frac{1}{i\omega} \left\{ \frac{1}{-i\omega} (1+2x) e^{-i\omega x} \Big|_{-1}^1 + \frac{1}{i\omega} \int_{-1}^1 2e^{-i\omega x} dx \right\} \quad (+2)$$

$$= \frac{2\hat{c}}{\omega} e^{-i\omega} + \frac{1}{\omega^2} (3e^{-i\omega} + e^{i\omega}) + \frac{1}{-\omega^2} \frac{2}{-i\omega} e^{-i\omega x} \Big|_{-1}^1$$

$$= \frac{2\hat{c}}{\omega} e^{-i\omega} + \frac{1}{\omega^2} (3e^{-i\omega} + e^{i\omega}) + \frac{2\hat{c}}{\omega^3} (e^{-i\omega} - e^{i\omega})$$

$$= \left(\frac{2\hat{c}}{\omega} + \frac{3}{\omega^2} - \frac{2\hat{c}}{\omega^3} \right) e^{-i\omega} + \left(\frac{1}{\omega^2} + \frac{2\hat{c}}{\omega^3} \right) e^{i\omega} \quad (+2)$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{2\hat{c}}{\omega} + \frac{3}{\omega^2} - \frac{2\hat{c}}{\omega^3} \right) e^{-i\omega} + \left(\frac{1}{\omega^2} + \frac{2\hat{c}}{\omega^3} \right) e^{i\omega} \right\} \quad (+1)$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (x+x^2) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (x+x^2) (\cos \omega x - i \sin \omega x) dx$$

(+1)

(+1)

$$\int_{-1}^1 (x+x^2) \sin \omega x dx = \int_{-1}^1 x \sin \omega x dx = \frac{1}{-\omega} x \cos \omega x \Big|_{-1}^1 + \frac{1}{\omega} \int_{-1}^1 \cos \omega x dx$$

$$= -\frac{2}{\omega} \cos \omega + \frac{1}{\omega^2} \sin \omega x \Big|_{-1}^1 = -\frac{2}{\omega} \cos \omega + \frac{2}{\omega^2} \sin \omega \quad (+3)$$

$$\int_{-1}^1 (x+x^2) \cos \omega x dx = \int_{-1}^1 x^2 \cos \omega x dx = \frac{1}{\omega} x^2 \sin \omega x \Big|_{-1}^1 - \frac{1}{\omega} \int_{-1}^1 2x \sin \omega x dx$$

$$= \frac{2}{\omega} \sin \omega - \frac{2}{\omega} \left[-\frac{2}{\omega} \cos \omega + \frac{2}{\omega^2} \sin \omega \right] = \frac{2}{\omega} \sin \omega + \frac{4}{\omega^2} \cos \omega - \frac{4}{\omega^3} \sin \omega$$

(+3)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{2}{\omega} \sin \omega + \frac{4}{\omega^2} \cos \omega - \frac{4}{\omega^3} \sin \omega \right) + i \left(\frac{2}{\omega} \cos \omega - \frac{2}{\omega^2} \sin \omega \right) \right\}$$