Quiz #4 (CSE 400.001)

Wednesday, November 13, 2013

1. (8 points) Find the Fourier series of the periodic function of period $p = 2L = 2$, $f(x) = x$ $(0 < x < 1)$, $f(x) = 0$ $(1 < x < 2)$, and $f(x) = f(x+2)$.

$$a_0 = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4} \quad +1$$

$$a_n = \int_0^1 x \cos n\pi x\,dx = \left[\frac{x}{n\pi}\sin n\pi x\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x\,dx$$

$$= \left(\frac{1}{n\pi}\right)^2 \left[\cos n\pi x\right]_0^1 = \frac{\cos n\pi - 1}{(n\pi)^2}$$

$$= \begin{cases} \dfrac{-2}{(n\pi)^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad +3$$

$$b_n = \int_0^1 x \sin n\pi x\,dx = \left[-\frac{x}{n\pi}\cos n\pi x\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx$$

$$= -\frac{1}{n\pi}\cos n\pi + \frac{1}{(n\pi)^2}\cdot\left[\sin n\pi x\right]_0^1$$

$$= \frac{(-1)^{n+1}}{n\pi} \quad +3$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$= \frac{1}{4} + \sum_{m=1}^{\infty} \frac{-2}{(2m-1)^2\pi^2}\cos(2m-1)\pi x$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{n\pi}\cdot\sin n\pi x \quad +1$$

2. (6 points) Find the Fourier transform of the function $f(x) = x^2e^{-x}$ (if $x > 0$), and $f(x) = 0$ (if $x < 0$).

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty x^2 e^{-x} \cdot e^{-i\omega x}\,dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty x^2 e^{-(1+i\omega)x}\,dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-1}{1+i\omega}x^2 e^{-(1+i\omega)x}\right]_0^\infty + \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{2}{(1+i\omega)} x e^{-(1+i\omega)x}\,dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{(1+i\omega)^2} x e^{-(1+i\omega)x}\right]_0^\infty + \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{2}{(1+i\omega)^2} e^{-(1+i\omega)x}\,dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{(1+i\omega)^3} e^{-(1+i\omega)x}\right]_0^\infty \quad +1$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{(1+i\omega)^3} \quad +1$$

3. (8 points) Find the cubic spline $g(x)$ to the following data, with $k_0 = 0$ and $k_3 = -6$:

$$f_0 = f(-1) = 1,\ f_1 = f(0) = 0,\ f_2 = f(1) = -1,\ f_3 = f(2) = 0.$$

+1

$$\begin{cases} k_0 + 4k_1 + k_2 = 3\cdot(-2) = -6 \\ k_1 + 4k_2 + k_3 = 3\cdot 0 = 0 \end{cases} \Rightarrow \begin{cases} 4k_1 + k_2 = -6 \\ k_1 + 4k_2 = 6 \end{cases}$$

$$k_1 + k_2 = 0,\quad k_1 - k_2 = -4 \Rightarrow k_1 = -2,\ k_2 = 2$$

+1

$$\begin{cases} p_0(x) = Ax^3 + Bx^2 - 2x, & (-1 \leqq x \leqq 0) \\ p_1(x) = ax^3 + bx^2 - 2x, & (0 \leqq x \leqq 1) \\ p_2(x) = \alpha(x-1)^3 + \beta(x-1)^2 + 2(x-1) - 1, & (1 \leqq x \leqq 2). \end{cases}$$

$$\begin{aligned} p_0(-1) &= -A + B + 2 = 1 \\ p_0'(-1) &= 3A - 2B - 2 = 0 \end{aligned} \Rightarrow \begin{cases} A = 0 \\ B = -1 \end{cases}$$

$$\therefore p_0(x) = -x^2 - 2x,\quad (-1 \leqq x \leqq 0)$$

+2

$$\begin{aligned} p_1(1) &= a + b - 2 = -1 \\ p_1'(1) &= 3a + 2b - 2 = 2 \end{aligned} \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

$$\therefore p_1(x) = 2x^3 - x^2 - 2x,\quad (0 \leqq x \leqq 1)$$

+2

$$\begin{aligned} p_2(2) &= \alpha + \beta + 2 - 1 = 0 \\ p_2'(2) &= 3\alpha + 2\beta + 2 = -6 \end{aligned} \Rightarrow \begin{cases} \alpha = -6 \\ \beta = 5 \end{cases}$$

$$\begin{aligned} \therefore p_2(x) &= -6(x-1)^3 + 5(x-1)^2 + 2(x-1) - 1 \\ &= -6x^3 + 23x^2 - 26x + 8, \end{aligned}$$

+2 $\quad (1 \leqq x \leqq 2)$

4. (3 points) Compute the following integral numerically using the Gauss quadrature with $n = 3$:

$$\int_0^1 \frac{dx}{1+x^2}$$

$$t = 2x - 1, \qquad x = \frac{1}{2}(t+1)$$

$$dx = \frac{1}{2}dt$$

+1

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{2}\int_{-1}^{1} \frac{1}{1+\frac{1}{4}(t+1)^2}\,dt$$

$$= \int_{-1}^{1} \frac{2}{4+(t+1)^2}\,dt$$

+1

$$\approx \frac{10}{9}\cdot\frac{1}{4+\left(1-\sqrt{\frac{3}{5}}\right)^2} + \frac{16}{9}\cdot\frac{1}{5}$$

$$+ \frac{10}{9}\cdot\frac{1}{4+\left(1+\sqrt{\frac{3}{5}}\right)^2}$$

+1

5. (10 points) Table 1 shows the result of applying the Improved Euler method to the following initial value problem:

$$y' = 0.5 - x + 2y, \quad y(0) = 1.$$

from $x = 0$ to $x = 1$ with $h = 0.2$ Fill in the blank and show your work for partial credit.

x_i	y_i
0.00	1.0000
0.20	1.5800
0.40	2.3904
0.60	3.5418
0.80	5.1979
1.00	7.6008

+2

Table 1: Improved Euler Method

$$k_1 = 0.2 f(x_n, y_n) = 0.2(0.5 - x_n + 2y_n)$$
$$= 0.1 - 0.2x_n + 0.4y_n \quad +1$$
$$y_{n+1}^{*} = y_n + k_1 = 0.1 - 0.2x_n + 1.4y_n \quad +2$$
$$k_2 = 0.2 f(x_{n+1}, y_{n+1}^{*})$$
$$= 0.2(0.5 - x_{n+1} + 2y_{n+1}^{*})$$
$$= 0.1 - 0.2(x_n + 0.2) + 0.4(0.1 - 0.2x_n + 1.4y_n)$$
$$= 0.1 - 0.28x_n + 0.56y_n \quad +3$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$
$$= y_n + \frac{1}{2}(0.2 - 0.48x_n + 0.96y_n)$$
$$= 0.1 - 0.24x_n + 1.48y_n \quad +2$$