Exercise #1

Consider a rotation about axis (1,1,1) by angle 60°.
What is the corresponding 3×3 rotation matrix?

<sol>

$$(a,b,c) = \frac{1}{\sqrt{3}}(1,1,1) \in S^2$$

$$2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

rotation angle in physical world

$$(\cos\theta, \sin\theta\,(a,b,c)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{3}}(1,1,1)\right)$$

$$\begin{bmatrix} x^2+w^2-y^2-z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & y^2+w^2-x^2-z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & w^2+z^2-x^2-y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}$$

$$w = \frac{\sqrt{3}}{2}, \quad x = y = z = \frac{1}{2\sqrt{3}}$$

$$2xy = 2yz = 2xz = \frac{1}{6}$$

$$2wx = 2wy = 2wz = \frac{1}{2}$$

$$2x^2 = 2y^2 = 2z^2 = \frac{1}{6}$$

$$\therefore R_q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

## Exercise #2

Given
$$R = \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} \in SO(3),$$

what is $q = (w, x, y, z) \in S^3$ s.t. $R_q = R$ ?

<sol>

$$r_{00} + r_{11} + r_{22} = 3 - 4(x^2 + y^2 + z^2)$$
$$= 3 - 4(1 - w^2) = 4w^2 - 1$$

$$\therefore w = \pm \frac{1}{2}\sqrt{1 + r_{00} + r_{11} + r_{22}}$$

$$\begin{cases} 4wz = r_{10} - r_{01} \\ 4wy = r_{02} - r_{20} \\ 4wx = r_{21} - r_{12} \end{cases} \Rightarrow \begin{cases} z = \frac{1}{4w}(r_{10} - r_{01}) \\ y = \frac{1}{4w}(r_{02} - r_{20}) \\ x = \frac{1}{4w}(r_{21} - r_{12}) \end{cases}$$

## Exercise #3

What other advantages quaternions have over 3x3 matrix representation?

<answer>

① Storage : 4 numbers instead of 9 numbers
$(w, x, y, z)$ $[r_{ij}]_{3\times 3}$

② Efficient multiplication : 16 instead of 27
$q_2 \cdot q_1$ $R_{q_2} \cdot R_{q_1}$

③ ....

④ --- Many other advantages as well as you come to know more about quaternions!