Engineering Mathematics I

(Comp 419.001)

Midterm Exam I: April 19, 1999

1. (10 points) A retired person has a sum $S(t)$ invested so as to draw interest at an annual rate r compounded continuously. Withdrawals for living expenses are made at a rate of k dollars per year; assume that the withdrawals are made continuously.

   (a) If the initial value of the investment is S_0, determine $S(t)$ at any time.

   $$S'(t) = rS(t) - k, \quad \frac{dS}{S - k/r} = rdt, \quad \ln(S - k/r) = rt + c, \quad S - k/r = \alpha e^{rt}$$
   $$S = \alpha e^{rt} + k/r, \quad S_0 = \alpha + k/r, \quad \alpha = (S_0 - k/r), \quad S(t) = (S_0 - k/r)e^{rt} + k/r$$

   (b) Assuming that S_0 and r are fixed, determine the withdrawal rate k_0 at which $S(t)$ will remain constant.

   $$S'(t) = 0, \quad r(S_0 - k_0/r)e^{rt} = 0, \quad rS_0 - k_0 = 0, \quad k_0 = rS_0$$

   (c) If k exceeds the value k_0, then $S(t)$ will decrease and ultimately become zero. Find the time T at which $S(T) = 0$.

   $$(S_0 - k/r)d^{rT} + k/r = 0, \quad e^{rT} = \frac{-k/r}{S_0 - k/r} = \frac{k}{k - rS_0}, \quad T = \frac{1}{r}\ln\left(\frac{k}{k - rS_0}\right)$$

2. (5 points) In the following equation, determine the value of b for which the equation is exact and then solve it using that value of b:

   $$(ye^{2xy} + x)\ dx + bxe^{2xy}\ dy = 0.$$

   $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad e^{2xy} + 2xye^{2xy} = be^{2xy} + 2bxye^{2xy}, \quad b = 1$$
   $$u = \int M dx = \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 + k(y), \quad \frac{\partial u}{\partial y} = xe^{2xy} + k'(y) = N = xe^{2xy}, \quad u(x,y) = \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 = c$$

3. (5 points) Solve the following initial value problem

   $$y' + xy = xy^{-1}, \quad y(0) = 2.$$

   Let $u = y^2$, then $u' = 2yy'$ and $u(0) = 4$. Thus, we have

   $$2yy' + 2xy^2 = 2x, \quad u' + 2xu = 2x, \quad h = \int 2xdx = x^2$$
   $$u = e^{-x^2}\left[\int e^{x^2} 2xdx + c\right] = e^{-x^2}[e^{x^2} + c] = 1 + ce^{-x^2}$$
   $$u(0) = 1 + c = 4, \quad c = 3, \quad u(x) = 1 + 3e^{-x^2}, \quad y(x) = \sqrt{1 + 3e^{-x^2}}$$

4. (10 points) A series circuit has a capacitor of 10^{-5} farad, a resistor of 3×10^2 ohms, and an inductor of 0.2 henry. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge on the capacitor at any time t.

$$
\begin{aligned}
10^5 Q + 300I + 0.2I' = 0, \quad Q(0) = 10^{-6} \\
10^5 Q + 300Q' + 0.2Q'' = 0, \quad Q'' + 1500Q' + 500000Q = 0, \\
(\lambda + 500)(\lambda + 1000) = 0, \quad Q = c_1 e^{-500t} + c_2 e^{-1000t}, \quad c_1 + c_2 = 10^{-6} \\
Q' = -500c_1 e^{-500t} - 1000c_2 e^{-1000t}, \quad Q'(0) = c_1 + 2c_2 = -I(0)/500 = 0 \\
c_1 = 2 \cdot 10^{-6}, c_2 = -10^{-6}, \quad Q(t) = 2 \cdot 10^{-6} e^{-500t} - 10^{-6} e^{-1000t}
\end{aligned}
$$

5. (5 points) Solve the following initial value problem

$$
y'' + 4y = 3 \sin 2x, \quad y(0) = 2, y'(0) = -1.
$$

$$
\begin{aligned}
\lambda^2 + 4 = 0, \quad \lambda = \pm 2i, \quad y_h = c_1 \cos 2x + c_2 \sin 2x, \quad y_p = c_3 x \cos 2x + c_4 x \sin 2x \\
y_p'' = -4c_3 \sin 2x - 4c_3 x \cos 2x + 4c_4 \cos 2x - 4c_4 x \sin 2x \\
y_p'' + 4y_p = -4c_3 \sin 2x + 4c_4 \cos 2x = 3 \sin 2x, \quad c_3 = -\frac{3}{4}, c_4 = 0 \\
y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4} x \cos 2x, \quad y(0) = c_1 = 2, \quad y'(0) = 2c_2 - \frac{3}{4} = -1, \quad c_2 = -\frac{1}{8} \\
y = 2 \cos 2x - \frac{1}{8} \sin 2x - \frac{3}{4} x \cos 2x
\end{aligned}
$$

6. (5 points) Solve the following initial value problem

$$
x^2 y'' + xy' - y = x \ln x, \quad y(1) = 0, y'(1) = 0.
$$

$$
\begin{aligned}
y = x^m, \quad m(m-1) + m - 1 = 0, \quad m^2 - 1 = 0, \quad m = \pm 1, \quad y_1 = x, y_2 = x^{-1} \\
y_h = c_1 x + c_2 x^{-1}, \quad W = -2x^{-1}, \quad r = \frac{\ln x}{x} \text{ since } y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \frac{\ln x}{x} \\
y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx = \frac{1}{4} x (\ln x)^2 - \frac{1}{4} x \ln x + \frac{1}{8} x \\
y = c_1 x + c_2 x^{-1} + \frac{1}{4} x (\ln x)^2 - \frac{1}{4} x \ln x + \frac{1}{8} x, \quad y(1) = c_1 + c_2 + \frac{1}{8} = 0 \\
y' = c_1 - c_2 x^{-2} + \frac{1}{4} (\ln x)^2 + \frac{1}{4} \ln x - \frac{1}{8}, \quad y'(1) = c_1 - c_2 - \frac{1}{8} = 0, \quad c_1 = 0, c_2 = -\frac{1}{8} \\
y = \frac{1}{8} x - \frac{1}{8} x^{-1} - \frac{x}{4} \ln x + \frac{x}{4} (\ln x)^2
\end{aligned}
$$

7. (10 points) The gamma function is defined as follows

$$
\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx.
$$

(a) Show that for $p > 0$

$$
\Gamma(p+1) = p\, \Gamma(p).
$$

$$
\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx = \left[-e^{-x} x^p \right]_0^\infty + p \int_0^\infty e^{-x} x^{p-1} dx = p\, \Gamma(p)
$$

(b) Show that $\Gamma(1) = 1$.

$$\Gamma(1) = \int_0^\infty e^{-x}x^0 dx = [-e^{-x}]_0^\infty = 1$$

(c) For a positive integer n, show that

$$\Gamma(n+1) = n!$$

$$\Gamma(n+1) = n\ \Gamma(n) = n(n-1)\ \Gamma(n-1) = \cdots = n!\Gamma(1) = n!$$

(d) Show that for $p > 0$

$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$

$$\begin{aligned}
\Gamma(p+n) &= (p+n-1)\ \Gamma(p+n-1) = (p+n-1)(p+n-2)\ \Gamma(p+n-2) \\
&= \cdots = (p+n-1)(p+n-2)\cdots p\Gamma(p) \\
\Gamma(p+n)/\Gamma(p) &= p(p+1)(p+2)\cdots(p+n-2)(p+n-1)
\end{aligned}$$

8. (5 points) Show that

$$F^{(n)}(s) = \mathcal{L}[(-t)^n f(t)].$$

$$F(s) = \int e^{-st}f(t)dt, \quad F'(s) = \int e^{-st}(-t)f(t)dt, \quad F''(s) = \int e^{-st}(-t)^2 f(t)dt$$

$$\text{Assume } F^{(k)}(s) = \int e^{-st}(-t)^k f(t)dt, \ \text{ then } F^{(k+1)}(s) = \int e^{-st}(-t)^{k+1} f(t)dt$$

9. (5 points) Show that

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+\omega^2)^2}\right) = \frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t).$$

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+\omega^2)^2}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\cdot\frac{s}{s^2+\omega^2}\right) = \cos\omega t * \cos\omega t = \int_0^t \cos\omega\tau\cdot\cos\omega(t-\tau)d\tau$$

$$= \frac{1}{2}\int_0^t [\cos\omega t + \cos\omega(2\tau - t)]\, d\tau = \frac{1}{2}t\cos\omega t + \frac{1}{2}\left[\frac{1}{2\omega}\sin\omega(2\tau-t)\right]_0^t = \frac{1}{2}t\cos\omega t + \frac{1}{2\omega}\sin\omega t$$

10. (10 points) Solve the following integral equation

$$y(t) = te^t - 2e^t\int_0^t e^{-\tau}y(\tau)d\tau$$

$$y(t) = te^t - 2\int_0^t e^{t-\tau}y(\tau)d\tau = te^t - 2y(t)*e^t, \quad Y(s) = \frac{1}{(s-1)^2} - 2Y(s)\frac{1}{s-1}$$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right), \quad y(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh t$$

11. (10 points) Solve the following initial value problem

$$y_1'' = y_1 + 3y_2, \quad y_1(0) = 2, y_1'(0) = 3,$$
$$y_2'' = 4y_1 - 4e^t, \quad y_2(0) = 1, y_2'(0) = 2.$$

$$(y_1 - y_2)'' = -3(y_1 - y_2) + 4e^t, \quad (y_1 - y_2)(0) = 1, (y_1 - y_2)'(0) = 1$$
$$s^2(Y_1 - Y_2) - s - 1 = -3(Y_1 - Y_2) + \frac{4}{s-1}$$
$$Y_1 - Y_2 = \frac{s+1}{s^2+3} + \frac{4}{(s^2+3)(s-1)} = \frac{1}{s-1}$$
$$y_1 - y_2 = e^t, \quad y_1 = y_2 + e^t$$
$$y_2'' = 4y_2, \quad s^2Y_2 - s - 2 = 4Y_2, \quad Y_2 = \frac{1}{s-2}, \quad y_2 = e^{2t}, y_1 = e^t + e^{2t}$$

12. (10 points) Solve the following initial value problem

$$\begin{aligned} y_1' + y_2 &= 2[1 - u(t - 2\pi)]\cos t, \\ y_1 + y_2' &= 0, \\ y_1(0) &= 0, \\ y_2(0) &= 1. \end{aligned}$$

13. (5 points) Find the inverse Laplace transform of the following function

$$\frac{s+1}{s^2}e^{-s}$$

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}e^{-s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-s}\right] = 1 \cdot u(t-1) + (t-1) \cdot u(t-1) = t \cdot u(t-1)$$

14. (5 points) Set up the model of the network in the following figure, assuming that all charges and currents are 0 when the switch is closed at $t = 0$.

$$\begin{aligned} 5i_1' + 20(i_1 - i_2) &= 60 \\ 20(i_2' - i_1') + 20i_2 + 30i_2' &= 0 \end{aligned}$$