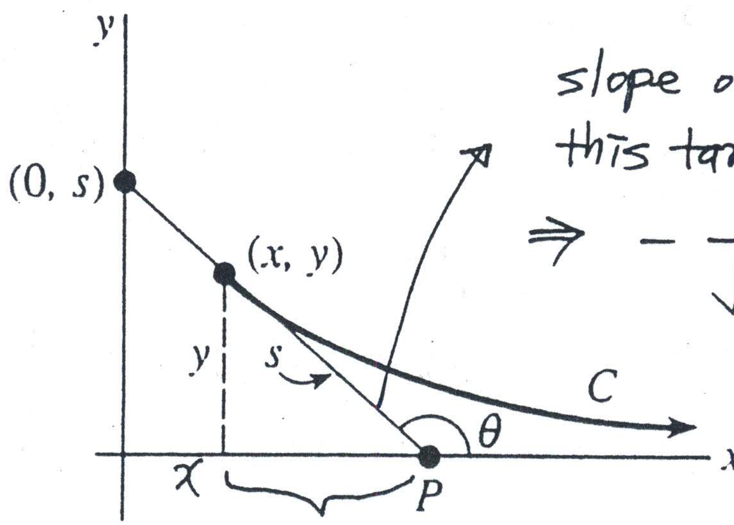
Engineering Mathematics I
(Comp 400.001)

Midterm Exam I: April 9, 2002

1. (15 points) A person P, starting at the origin, moves in the direction of the positive x-axis, pulling a weight along the curve C. The weight, initially located on the y-axis at $(0, s)$, is pulled by a rope of constant length s, which is kept tight throughout the motion. Find a differential equation that describes the curve C. Assume that the rope is always tangent to C.



slope of this tangent line $\Rightarrow -\dfrac{y}{\sqrt{s^2 - y^2}}$ +5

$\Downarrow$

length $= \sqrt{s^2 - y^2}$ +5

$$\therefore\ y' = -\frac{y}{\sqrt{s^2 - y^2}}$$ +5

2. (20 points) Find the orthogonal trajectory of the following family of curves, where c is arbitrary.

$$(x-c)^2 + y^2 = c^2$$

$$2(x-c) + 2y \cdot y' = 0 \quad \Rightarrow \quad y' = \frac{c-x}{y}$$

Let $y' = \frac{y}{x-c}$ (+5)

Since $x^2 - 2cx + y^2 = 0$, $\quad c = \frac{x^2+y^2}{2x}$

$$\Rightarrow y' = \frac{y}{x - \frac{x^2+y^2}{2x}} = \frac{2xy}{x^2-y^2} \quad \text{(+5)}$$

$$= \frac{2(y/x)}{1-(y/x)^2}$$

Let $u = y/x$, $\quad y = ux$

$$y' = u'x + u$$

$$u'x + u = \frac{2u}{1-u^2}, \qquad u'x = \frac{u+u^3}{1-u^2}$$

$$\frac{1}{x}dx = \frac{1-u^2}{u+u^3}du = \left(\frac{1}{u} - \frac{2u}{1+u^2}\right)du$$

$$\ln x = \ln u - \ln(1+u^2) + \hat{c}$$

$$x = e^{\hat{c}} \cdot \frac{u}{1+u^2} = \tilde{c} \cdot \frac{y/x}{1+(y/x)^2} = \tilde{c}\,\frac{xy}{x^2+y^2},$$

$$x^2 + y^2 - \tilde{c}y = 0$$

$$x^2 + (y-c^*)^2 = (c^*)^2, \quad \text{where } c^* = \frac{1}{2}\tilde{c} = \frac{1}{2}e^{\hat{c}}$$

(+10)

3. (15 points) Solve the following differential equation

$$x^2y'' - 3xy' + 13y = 4 + 3x, \ x > 0.$$

$$m(m-1) - 3m + 13 = 0$$

$$m^2 - 4m + 13 = 0$$

$$(m-2)^2 + 3^2 = 0$$

$$m = 2 \pm 3i$$

+3

$$y_h = x^2\left[A\cos(3\ln x) + B\sin(3\ln x)\right]$$

+4

$$y_p = c_1 + c_2 x$$

$$y_p' = c_2$$

$$y_p'' = 0$$

+3

$$-3x(c_2) + 13(c_1 + c_2 x) = 4 + 3x$$

$$13c_1 + 10c_2 x = 4 + 3x$$

$$c_1 = \frac{4}{13}, \quad c_2 = \frac{3}{10}$$

+3

$$\therefore y = x^2\left[A\cos(3\ln x) + B\sin(3\ln x)\right] + \frac{4}{13} + \frac{3}{10}x$$

+2

4. (20 points) Solve the following initial value problem

$$y' + y = r(t), \quad y(0) = 5,$$

where

$$r(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 3\sin t & \text{if } t \ge \pi. \end{cases}$$

$$\left.\begin{aligned} y' + y &= 3\sin t \cdot u(t-\pi) \\ &= -3\sin(t-\pi)\cdot u(t-\pi) \end{aligned}\right] +5$$

$$\left. sY - 5 + Y = -3e^{-\pi s}\cdot\frac{1}{s^2+1} \right] +5$$

$$(s+1)Y = 5 - 3e^{-\pi s}\cdot\frac{1}{s^2+1}$$

$$\left.\begin{aligned} Y &= \frac{5}{s+1} - 3e^{-\pi s}\cdot\frac{1}{(s+1)(s^2+1)} \\ &= \frac{5}{s+1} - \frac{3}{2}e^{-\pi s}\cdot\left[\frac{1}{s+1} - \frac{s-1}{s^2+1}\right] \\ &= \frac{5}{s+1} - \frac{3}{2}e^{-\pi s}\cdot\left[\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] \end{aligned}\right] +5$$

$$y(t) = 5e^{-t} - \frac{3}{2}\left[e^{-(t-\pi)} - \cos(t-\pi) + \sin(t-\pi)\right]u(t-\pi)$$

$$= \begin{cases} 5e^{-t} & \text{if } 0 \le t < \pi \\ 5e^{-t} - \frac{3}{2}\left[e^{-(t-\pi)} + \cos t - \sin t\right] & \text{if } t \ge \pi \end{cases}$$

+5

5. (15 points) Solve the following integral equation

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau$$

$$f(t) = 3t^2 - e^{-t} - f(t) * e^t \quad +5$$

$$F(s) = 3\cdot\frac{2!}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1} \quad +3$$

$$\frac{s}{s-1}\cdot F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$F(s) = \frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)}$$

$$= 3\cdot\frac{2!}{s^3} - \frac{3!}{s^4} + \left[\frac{1}{s} - \frac{2}{s+1}\right] \quad +3$$

$$f(t) = 3\cdot t^2 - t^3 + 1 - 2e^{-t} \quad +4$$

6. (15 points) Solve the following initial value problem

$$y_1'' = -10y_1 + 4y_2, \quad y_1(0) = 0, y_1'(0) = 1,$$
$$y_2'' = 4y_1 - 4y_2, \quad y_2(0) = 0, y_2'(0) = -1.$$

$$\left\{\begin{array}{l} s^2 Y_1 - s\,\overset{0}{\cancel{y_1(0)}} - y_1'(0) = -10Y_1 + 4Y_2 \\ s^2 Y_2 - s\,\underset{0}{\cancel{y_2(0)}} - y_2'(0) = 4Y_1 - 4Y_2 \end{array}\right] \quad +5$$

$$\left\{\begin{array}{l} (s^2+10)Y_1 - 4Y_2 = 1 \\ -4Y_1 + (s^2+4)Y_2 = -1 \end{array}\right.$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{(s^2+10)(s^2+4)-16}\begin{bmatrix} s^2+4 & 4 \\ 4 & s^2+10 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{(s^2+2)(s^2+12)}\begin{bmatrix} s^2 \\ -s^2-6 \end{bmatrix} \quad +5$$

$$\left\{\begin{array}{l} Y_1 = -\frac{1}{5\sqrt{2}}\,\frac{\sqrt{2}}{s^2+2} + \frac{6}{5\sqrt{12}}\cdot\frac{\sqrt{12}}{s^2+12} \\ Y_2 = -\frac{2}{5\sqrt{2}}\cdot\frac{\sqrt{2}}{s^2+2} - \frac{3}{5\sqrt{12}}\cdot\frac{\sqrt{12}}{s^2+12} \end{array}\right.$$

$$\left\{\begin{array}{l} y_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin\sqrt{12}t \\ y_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin\sqrt{12}t \end{array}\right] \quad +5$$