Engineering Mathematics I
(Comp 400.001)

Midterm Exam I: October 18, 2010

1. (20 points) If a certain business were operated on a nonprofit basis, its value $v(t)$ at time t would increase at a rate proportional to that value. The managers of the business decide, however, to reduce its value at a rate equal to a fixed fraction of the value to obtain a profit $p(t)$ from the business.

   (a) (8 points) Find a pair of differential equations that can be used to compute $v(t)$ and $p(t)$.

   (b) (6 points) Find $v(t)$ with an initial condition: $v(0) = v_0$.

   (c) (6 points) Find $p(t)$ with an initial condition $p(0) = 0$.

(a) $$\begin{cases} v'(t) = k_1 (v(t) - p'(t)) \\ p'(t) = k_2 v(t) \end{cases}$$

$$\Rightarrow \begin{cases} v'(t) = k_1 (1-k_2) v(t) \\ p'(t) = k_2 v(t) \end{cases}$$

(b) $$v(t) = v_0 \cdot e^{k_1(1-k_2)t}$$

(c) $$p(t) = \frac{k_2 v_0}{k_1(1-k_2)} \left[e^{k_1(1-k_2)t} - 1 \right]$$

2. (15 points) Show that if $p(x)$ is a polynomial of degree n, then

$$y' + ay = p(x)$$

has a solution that is a polynomial of degree n whenever $a \neq 0$.

$$p(x) = \sum_{k=0}^{n} b_k x^k, \quad \text{where } b_n \neq 0 \quad +1$$

Let $y(x) = \sum_{k=0}^{n} c_k x^k$, then $+1$

$$y'(x) = \sum_{k=1}^{n} k \cdot c_k x^{k-1} = \sum_{k=0}^{n-1} (k+1) c_{k+1} x^k \quad +1$$

$$y' + ay = a \cdot c_n + \sum_{k=0}^{n-1} \left[(k+1) c_{k+1} + a c_k \right] x^k$$

$$= \sum_{k=0}^{n} b_k \cdot x^k = p(x) \quad +2$$

$a \cdot c_n = b_n$, $(k+1) c_{k+1} + a c_k = b_k$, $+1$

for $k = 0, \cdots, n-1$

$$\Rightarrow \begin{cases} c_n = \frac{1}{a} \cdot b_n \neq 0 \quad +2 \\ c_k = \frac{1}{a}\left[b_k - (k+1) c_{k+1} \right], \text{ for } k = n-1, \cdots, 0. \end{cases}$$

$+1$

$\therefore y(x) = \sum_{k=0}^{n} c_k x^k$ is a solution that is a polynomial of degree n.

$+1$

3. (15 points) Solve the following initial value problem

$$xy'' + y' = x, \quad y(1) = 1,\ y'(1) = 1.$$

$x^2 y'' + x y' = x^2$ +2

$m(m-1) + m = 0$, $m^2 = 0$ +1

$y_1 = x^0 = 1$, $y_2 = x^0 \ln x = \ln x$ +2

$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$, $W_1 = -\ln x$, $W_2 = 1$

+1 +1

$y'' + \frac{1}{x} y' = \overbrace{1}^{=r(x)}$ +2

$y_p = y_1 \cdot \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx$

$= \int (-x \ln x)\,dx + (\ln x) \int x\,dx$

$= -\cancel{\frac{1}{2}x^2 \ln x} + \frac{1}{4}x^2 + \cancel{\frac{1}{2}x^2 \ln x}$

$= \frac{1}{4}x^2$ +2

$y = c_1 + c_2 \ln x + \frac{1}{4}x^2$ +1

$y(1) = c_1 + \frac{1}{4} = 1 \Rightarrow c_1 = \frac{3}{4}$

$y'(1) = c_2 + \frac{1}{2} = 1 \Rightarrow c_2 = \frac{1}{2}$

+2

$\therefore y = \frac{3}{4} + \frac{1}{2}\ln x + \frac{1}{4}x^2$

+1

4. (10 points) Solve the following equation

$$y''' - 3y' - 2y = \sin x + \cos x.$$

$$\lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2) = 0 \quad +1$$

$$y_1 = e^{-x}, \quad y_2 = xe^{-x}, \quad y_3 = e^{2x} \quad +2$$

$$y_p = A\cos x + B\sin x$$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x \quad +2$$

$$y_p''' = A\sin x - B\cos x$$

$$y_p''' - 3y_p' - 2y_p$$

$$= (-2A-4B)\cos x + (4A-2B)\sin x \quad +2$$

$$= \sin x + \cos x$$

$$\Rightarrow \quad 4A - 2B = 1, \quad -2A - 4B = 1$$

$$A = 0.1, \quad B = -0.3 \quad +2$$

$$\therefore y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{2x} \quad +1$$

$$+ 0.1\cos x - 0.3\sin x$$

5. (20 points) Solve the following system of ODEs:

$$y_1' = y_1 + 3y_2 + \sin x$$
$$y_2' = y_1 - y_2 - \cos x$$

$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$, $\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3$

$= \lambda^2 - 4 = 0$

$\therefore \lambda_1 = -2,\ \lambda_2 = 2$ +2

$\lambda_1 = -2,\ x^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 2,\ x^{2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ +3

$y_h = c_1 e^{-2x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ +2

$y_p = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$ +3

$y_p' = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix} = \begin{bmatrix} -B & A \\ -D & C \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$ +3

$\begin{bmatrix} -B & A \\ -D & C \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$

$\begin{bmatrix} -B & A \\ -D & C \end{bmatrix} = \begin{bmatrix} A+3C+1 & B+3D \\ A-C & B-D-1 \end{bmatrix}$ +3

$A = -0.2,\ B = 0.4,\ C = -0.4,\ D = -0.2$ +2

$\therefore y_1 = c_1 e^{-2x} + 3c_2 e^{2x} - 0.2 \sin x + 0.4 \cos x$

$y_2 = -c_1 e^{-2x} + c_2 e^{2x} - 0.4 \sin x - 0.2 \cos x$ +2

6. (20 points) The following Table compares the results of applying the Euler, improved Euler, and Runge-Kutta methods to the following initial value problem with $h = 0.2$:

$$y' = 4x + 2y, \quad y(0) = 0.$$

Fill in the three blanks in (A), (B), and (C), and show your work for partial credit.

x_i	Euler	Improved Euler	Runge-Kutta	Exact
0.0	0.0000	0.0000	0.0000	0.0000
0.2	0.0000	0.0800	(C)	0.0918
0.4	0.1600	(B)	0.4253	0.4255
0.6	(A)	1.0418	1.1195	1.1201
0.8	1.2416	2.1979	2.3518	2.3530

(A) $k_1 = hf(x_2, y_2) = 0.2[4\times 0.4 + 2\times 0.1600] = 0.3840$ (+2)

$\therefore y_3 = y_2 + k_1 = \underline{0.5440}$ (+2)

(B) $k_1 = hf(x_1, y_1) = 0.2[4\times 0.2 + 2\times 0.0800] = 0.1920$ (+2)

$k_2 = hf(x_2, y_1 + k_1) = 0.2[4\times 0.4 + 2\times 0.2720] = 0.4288$ (+2)

$\therefore y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = \underline{0.3904}$ (+2)

(C) $k_1 = hf(x_0, y_0) = 0.2[4\times 0.0 + 2\times 0.0000] = 0.0000$ (+2)

$k_2 = hf(x_0 + 0.1, y_0 + 0.5k_1)$

$= 0.2[4\times 0.1 + 2\times 0.0000] = 0.0800$ (+2)

$k_3 = hf(x_0 + 0.1, y_0 + 0.5k_2)$

$= 0.2[4\times 0.1 + 2\times 0.0400] = 0.0960$ (+2)

$k_4 = hf(x_1, y_0 + k_3)$

$= 0.2[4\times 0.2 + 2\times 0.0960] = 0.1984$ (+2)

$\therefore y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \underline{0.0917}$

(+2)