Quiz #2 (CSE 400.001)

Wednesday, October 5, 2011

1. (5 points) Solve the following equation:

$$y' - 2y = x^2 e^{2x}.$$

Solution:

$$\begin{aligned} y(x) &= e^{-\int(-2)dx}\left[\left(\int e^{\int(-2)dx} x^2 e^{2x}\right) dx + C\right] \quad +3 \\ &= e^{2x}\left[\left(\int e^{-2x} x^2 e^{2x}\right) dx + C\right] \\ &= e^{2x}\left[\left(\int x^2\right) dx + C\right] \\ &= Ce^{2x} + \frac{1}{3}x^3 e^{2x} \end{aligned}$$

+2

2. (10 points) Find a particular solution of

$$y'' - 4y' + 4y = 4e^{2x} + \sin x + 7\cos x.$$

Solution:

Let $y_p = y_{p_1} + y_{p_2}$ with $y_{p_1} = Cx^2e^{2x}$ and $y_{p_2} = A\cos x + B\sin x$, then

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$$\begin{aligned} y_{p_1}(x) &= Cx^2e^{2x} \\ y'_{p_1}(x) &= 2Cxe^{2x} + 2Cx^2e^{2x} \\ y''_{p_1}(x) &= 2Ce^{2x} + 8Cxe^{2x} + 4Cx^2e^{2x} \\ y''_{p_1}(x) - 4y'_{p_1}(x) + 4y_{p_1}(x) &= 2Ce^{2x} = 4e^{2x} \\ y_{p_1}(x) &= 2x^2e^{2x} \end{aligned}$$

+2

$$\begin{aligned} y_{p_2}(x) &= A\cos x + B\sin x \\ y'_{p_2}(x) &= -A\sin x + B\cos x \\ y''_{p_2}(x) &= -A\cos x - B\sin x \\ y''_{p_2}(x) - 4y'_{p_2}(x) + 4y_{p_2}(x) &= (3A - 4B)\cos x + (4A + 3B)\sin x \\ &= 7\cos x + \sin x \\ y_{p_2}(x) &= \cos x - \sin x \end{aligned}$$

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$$y_p = y_{p_1} + y_{p_2} = 2x^2e^{2x} + \cos x - \sin x$$

3. (10 points) When $y_1(x)$ and $y_2(x)$ form a basis of solutions of the following equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

show that

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(x)}dx, \quad \text{with } W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x),$$

is a particular solution for the following nonhomogeneous linear ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x).$$

Proof: Let $u_1(x) = -\int \frac{y_2(x)r(x)}{W(x)}dx$ and $u_2(x) = \int \frac{y_1(x)r(x)}{W(x)}dx$, then

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x), \\ y_p'(x) &= u_1'(x)y_1(x) + u_2'(x)y_2(x) + u_1(x)y_1'(x) + u_2(x)y_2'(x), \end{aligned}$$

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where

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = -\frac{y_2(x)r(x)}{W(x)}y_1(x) + \frac{y_1(x)r(x)}{W(x)}y_2(x) = 0.$$

+3

Thus, we have

$$\begin{aligned} y_p'(x) &= u_1(x)y_1'(x) + u_2(x)y_2'(x), \\ y_p''(x) &= u_1'(x)y_1'(x) + u_2'(x)y_2'(x) + u_1(x)y_1''(x) + u_2(x)y_2''(x). \end{aligned}$$

+2

Consequently,

$$\begin{aligned} & y_p''(x) + p(x)y_p'(x) + q(x)y_p(x) \\ =\ & u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \\ & + u_1(x)\left[y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\right] \\ & + u_2(x)\left[y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)\right] \\ =\ & u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \\ =\ & -\frac{y_2(x)r(x)}{W(x)}y_1'(x) + \frac{y_1(x)r(x)}{W(x)}y_2'(x) \\ =\ & \frac{y_1(x)y_2'(x) - y_2(x)y_1'(x)}{W(x)}r(x) \\ =\ & r(x). \end{aligned}$$

+3