Engineering Mathematics I
(Comp 400.001)

Midterm Exam, October 31, 2012

< Solution >

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

1. (20 points) Consider Newton's law of cooling: $dT/dt = k(T - T_m)$, $k < 0$, where the temperature of the surrounding medium T_m changes with time. Suppose the initial temperature of a body is T_1 and the initial temperature of the surrounding medium is T_2 and $T_m = T_2 + B(T_1 - T)$, where $B > 0$ is a constant.

(a) (15 points) Find the temperature of the body at any time t.

(b) (3 points) What is the limiting value of the temperature as $t \to \infty$?

(c) (2 points) What is the limiting value of T_m as $t \to \infty$?

(a) $T - T_m = (1+B)T - (T_2 + BT_1)$ +3

$$\frac{dT}{(1+B)T - (T_2+BT_1)} = k\,dt$$

$$\ln\left|T - \frac{T_2+BT_1}{1+B}\right| = k(1+B)t + C^*$$ +3

$$T = \frac{T_2+BT_1}{1+B} + C\cdot e^{k(1+B)t}$$ +3

$$T_1 = \frac{T_2+BT_1}{1+B} + C$$

$$\therefore C = \frac{T_1 - T_2}{1+B}$$ +3

$$T = \frac{T_2+BT_1}{1+B} + \frac{T_1-T_2}{1+B} e^{k(1+B)t}$$ +3

(b) $$\lim_{t\to\infty} T = \frac{T_2+BT_1}{1+B}$$

(c) $$\lim_{t\to\infty} T_m = T_2 + BT_1 - B\cdot\frac{T_2+BT_1}{1+B}$$

$$= \frac{T_2+BT_1}{1+B}$$

2. (10 points) When $y_1(x)$ and $y_2(x)$ form a basis of solutions of the following equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

show that

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(x)}dx, \quad \text{with } W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x),$$

is a particular solution for the following nonhomogeneous linear ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x).$$

Proof: Let $u_1(x) = -\int \frac{y_2(x)r(x)}{W(x)}dx$ and $u_2(x) = \int \frac{y_1(x)r(x)}{W(x)}dx$, then +2

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x), \\ y_p'(x) &= u_1'(x)y_1(x) + u_2'(x)y_2(x) + u_1(x)y_1'(x) + u_2(x)y_2'(x), \end{aligned}$$ +1

where

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = -\frac{y_2(x)r(x)}{W(x)}y_1(x) + \frac{y_1(x)r(x)}{W(x)}y_2(x) = 0.$$ +3

Thus, we have

$$\begin{aligned} y_p'(x) &= u_1(x)y_1'(x) + u_2(x)y_2'(x), \\ y_p''(x) &= u_1'(x)y_1'(x) + u_2'(x)y_2'(x) + u_1(x)y_1''(x) + u_2(x)y_2''(x). \end{aligned}$$ +1

Consequently,

$$\begin{aligned} & y_p''(x) + p(x)y_p'(x) + q(x)y_p(x) \\ = \; & u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \\ & + u_1(x)\left[y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\right] \\ & + u_2(x)\left[y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)\right] \\ = \; & u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \\ = \; & -\frac{y_2(x)r(x)}{W(x)}y_1'(x) + \frac{y_1(x)r(x)}{W(x)}y_2'(x) \\ = \; & \frac{y_1(x)y_2'(x) - y_2(x)y_1'(x)}{W(x)}r(x) \\ = \; & r(x). \end{aligned}$$ +3

3. (10 points) Solve the following initial value problem:

$$4x^2y'' + y = 0, \quad y(1) = 2, y'(1) = 4.$$

$$4m(m-1)+1=0$$

$$4m^2-4m+1=0$$

$$\left(m-\frac{1}{2}\right)^2=0$$

+2

$$y = c_1\sqrt{x} + c_2\sqrt{x}\ln x$$ +2

$$c_1 = 2$$ +2

$$y' = c_1\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}} + c_2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\ln x$$

$$+ c_2\sqrt{x}\cdot\frac{1}{x}$$

$$\frac{1}{2}\cdot 2 + c_2 = 4$$

$$\therefore c_2 = 3$$

+3

$$\therefore y = 2\sqrt{x} + 3\sqrt{x}\ln x$$

+1

4. (15 points) Find the general solution of the following equation:

$$y'' - 2y' + y = x^2 + 6xe^x.$$

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$
(+2)

$$y_h = c_1 e^x + c_2 x e^x$$ (+1)

$$y_p = y_{p_1} + y_{p_2}$$

$$y_{p_1} = A + Bx + Cx^2$$ (+2)

(+2)
$$y_{p_1}' = B + 2Cx$$
$$y_{p_1}'' = 2C$$
$$y_{p_1}'' - 2y_{p_1}' + y_{p_1} = (A - 2B + 2C) + (B - 4C)x + Cx^2$$
$$= x^2$$
$$\therefore A = 6,\ B = 4,\ C = 1$$

$$y_{p_2} = Dx^3 e^x$$ (+5)

$$y_{p_2}' = D(3x^2 + x^3)e^x$$
$$y_{p_2}'' = D(6x + 6x^2 + x^3)e^x$$
$$y_{p_2}'' - 2y_{p_2}' + y_{p_2} = 6Dxe^x = 6xe^x$$
$$\therefore D = 1$$
(+2)

$$\therefore y = y_h + y_{p_1} + y_{p_2} = c_1 e^x + c_2 x e^x + 6 + 4x + x^2 + x^3 e^x$$

(+1)

5. (20 points) Solve the following initial value problem

$$y'' - 5y' + 6y = 2e^t u(t-1) + \delta(t-2), \quad y(0) = 0, \ y'(0) = 1.$$

$$y'' - 5y' + 6y = \underline{2e \cdot e^{t-1} u(t-1)} + \delta(t-2)$$

+3

$$s^2 Y - 1 - 5sY + 6Y = 2e \cdot e^{-s} \cdot \frac{1}{s-1} + e^{-2s}$$

+3

$$(s-2)(s-3)Y = 1 + 2 \cdot \frac{e^{1-s}}{s-1} + e^{-2s}$$

$$Y = \frac{1 + e^{-2s}}{(s-2)(s-3)} + 2 \cdot \frac{e^{1-s}}{(s-1)(s-2)(s-3)}$$

+3

$$= (1 + e^{-2s})\left(\frac{1}{s-3} - \frac{1}{s-2}\right)$$

$$+ e^{1-s} \cdot \left(\frac{1}{s-3} - \frac{2}{s-2} + \frac{1}{s-1}\right)$$

+3

$$y(t) = \left(e^{3t} - e^{2t}\right)$$ +2

$$+ \left(e^{3(t-2)} - e^{2(t-2)}\right) u(t-2)$$

+6

$$+ e \cdot \left(e^{3(t-1)} - 2e^{2(t-1)} + e^{t-1}\right) u(t-1)$$

6. (15 points) Solve the following equation

$$f'(t) = 1 - \sin t - \int_0^t f(\tau)d\tau, \quad f(0) = 0$$

$$s F(s) = \frac{1}{s} - \frac{1}{s^2+1} - \frac{1}{s} \cdot F(s) \quad +3$$

$$\frac{s^2+1}{s} F(s) = \frac{1}{s} - \frac{1}{s^2+1}$$

$$F(s) = \frac{1}{s^2+1} - \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \quad +2$$

$$f(t) = \sin t - (\sin t) * (\cos t) \quad +3$$

$$= \sin t - \int_0^t \sin \tau \cdot \cos(t-\tau) d\tau$$

$$= \sin t - \frac{1}{2}\int_0^t \sin t \, d\tau$$

$$- \frac{1}{2}\int_0^t \sin(2\tau - t) d\tau$$

$$+3$$

$$= \sin t - \frac{1}{2} t \sin t$$

$$- \frac{1}{2} \cdot \frac{1}{2} \left[-\cos(2\tau - t) \right]_0^t \quad +2$$

$$= \sin t - \frac{1}{2} t \sin t \quad +2$$

7. (10 points) Find the inverse Laplace transform of the following function:

$$\ln \frac{s^2 + 2s + 5}{s^2 + 4s + 5}$$

$$F(s) = \ln(s^2+2s+5) - \ln(s^2+4s+5) \quad +3$$

$$F'(s) = \frac{2(s+1)}{(s+1)^2+2^2} - \frac{2(s+2)}{(s+2)^2+1^2} \quad +3$$

$$-tf(t) = 2 \cdot e^{-t}\cos 2t - 2e^{-2t}\cos t \quad +3$$

$$\therefore f(t) = \frac{2}{t}\left(e^{-2t}\cos t - e^{-t}\cos 2t\right)$$

+1