Chap 3. Higher Order Linear ODEs

3.1 Homogeneous Linear ODEs

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x)\, y' + P_0(x)\, y = 0$$

Th 1 (Fundamental Th for the Homogeneous Linear ODE)

The solution space for a homogeneous linear ODE of order n is a vector space of dimension n

a general solution on an open interval I is of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $y_1(x), y_2(x), \cdots, y_n(x)$ is a basis of the solution space.

Basis, linear dependence/independence, existence and uniqueness, Wronskian, and initial value problem — These are about the same as the second order case.

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1, & \cdots, & y_n \\ y_1', & \cdots, & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{vmatrix}$$

Ex 3: $y'''' - 5y'' + 4y = 0$

$$\lambda^4 - 5\lambda^2 + 4 = (\lambda^2 - 4)(\lambda^2 - 1) = (\lambda+2)(\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

Ex 4:

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0$$

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = (m-1)(m-2)(m-3) = 0$$

$$\therefore y = c_1 x + c_2 x^2 + c_3 x^3$$

3.2 Homogeneous Linear ODEs with Const. Coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

$$\lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

Distinct Real Roots : $\lambda_1, \cdots, \lambda_n$

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}$$

$$W(e^{\lambda_1 x}, \cdots, e^{\lambda_n x}) = \begin{vmatrix} e^{\lambda_1 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & & \lambda_n e^{\lambda_n x} \\ \vdots & & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= e^{(\lambda_1 + \cdots + \lambda_n)x} \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

Vandermonde or Cauchy Determinant

$$= e^{(\lambda_1 + \cdots + \lambda_n)x} \cdot (-1)^{\frac{n(n-1)}{2}} \prod_{j<k} (\lambda_j - \lambda_k)$$

Simple Complex Roots

$$e^{\gamma x} \cos \omega x, \quad e^{\gamma x} \sin \omega x$$

Multiple Real Roots

λ: root of order m $\Rightarrow$ $e^{\lambda x}, x e^{\lambda x}, \cdots, x^{m-1} e^{\lambda x}$

Multiple Complex Roots

$$e^{\lambda x} \cos \omega x, \; x e^{\lambda x} \cos \omega x, \; \cdots, \; x^{m-1} e^{\lambda x} \cos \omega x$$
$$e^{\lambda x} \sin \omega x, \; x e^{\lambda x} \sin \omega x, \; \quad , \; x^{m-1} e^{\lambda x} \sin \omega x$$

Ex 1

$$y''' - 2y'' - y' + 2y = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$$

Ex 2

$$y''' - y'' + 100y' - 100y = 0, \quad y(0) = 4, \; y'(0) = 11, \; y''(0) = -299$$

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = (\lambda - 1)(\lambda^2 + 100) = 0$$

$$y = c_1 e^{x} + A\cos 10x + B\sin 10x$$

$$y' = c_1 e^{x} - 10A\sin 10x + 10B\cos 10x$$

$$y'' = c_1 e^{x} - 100A\cos 10x - 100B\sin 10x$$

$$\begin{cases} c_1 + A = 4 \\ c_1 + 10B = 11 \\ c_1 - 100A = -299 \end{cases} \Rightarrow c_1 = 1, \; A = 3, \; B = 1$$

$$\therefore y = e^{x} + 3\cos 10x + \sin 10x$$

Ex 3

$$y^{(5)} - 3y^{(4)} + 3y^{(3)} - y^{(2)} = 0$$

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = \lambda^2(\lambda - 1)^3 = 0$$

$$\therefore y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2)e^{x}$$

3.3 Nonhomogeneous Linear ODEs

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = r(x)$$

$$y(x) = y_h(x) + y_p(x)$$

IVP : $y(x_0) = K_0, \; y'(x_0) = K_1, \; \cdots, \; y^{(n-1)}(x_0) = K_{n-1}$

Method of Undetermined Coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$$

Ex 1

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \quad y(0)=3, \quad y'(0)=-3, \quad y''(0)=-47$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3 = 0$$

$$y_h = (c_1 + c_2 x + c_3 x^2) e^{-x}$$

$$\begin{cases} y_p = c x^3 e^{-x} \\ y_p' = c(3x^2 - x^3) e^{-x} \\ y_p'' = c(6x - 6x^2 + x^3) e^{-x} \\ y_p''' = c(6 - 18x + 9x^2 - x^3) e^{-x} \end{cases}$$

$$\Rightarrow c = 5$$

$$y_p = 5x^3 e^{-x}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + 5x^3 e^{-x}$$

$$y(0) = c_1 = 3$$

$$y' = (c_2 + 2c_3 x - 3 - c_2 x - c_3 x^2) e^{-x} + (15x^2 - 5x^3) e^{-x}$$

$$y'(0) = c_2 - 3 = -3 \quad \therefore c_2 = 0$$

$$\therefore y' = (-3 + 2c_3 x - c_3 x^2) e^{-x} + (15x^2 - 5x^3) e^{-x}$$

$$y'' = (2c_3 - 2c_3 x + 3 - 2c_3 x + c_3 x^2) e^{-x}$$
$$+ (30x - 15x^2 - 15x^2 + 5x^3) e^{-x}$$

$$y''(0) = 2c_3 + 3 = -47 \quad \therefore c_3 = -25$$

$$\therefore y = (3 - 25x^2) e^{-x} + 5x^3 e^{-x}$$

Method of Variation of Parameters

$$y_p(x) = \sum_{j=1}^{n} y_j(x) \cdot \int \frac{W_j(x)}{W(x)} r(x)\,dx$$

$$W_j = \begin{vmatrix} y_1 & \cdots & \cdots & y_n \\ y_1^{(1)} & \cdots & \cdots & y_n^{(1)} \\ \vdots & & & \vdots \\ y_1^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$\hookrightarrow$ Replace the j-th column by $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Ex 3 (Nonhomogeneous Euler-Cauchy Equation)

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = x^4 \ln x \quad (x > 0)$$

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = (m-1)(m-2)(m-3) = 0$$

$$\therefore y_1 = x,\ y_2 = x^2,\ y_3 = x^3$$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = -2x^3, \quad W_3 = x^2$$

Now, $y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3}y = \overbrace{x\ln x}^{=r(x)}$

$$y_p = x\int \frac{x^4}{2x^3} \cdot x\ln x\,dx + x^2 \int \frac{(-2x^3)}{2x^3} x\ln x\,dx + x^3 \int \frac{x^2}{2x^3} x\ln x\,dx$$

$$= x\int \frac{1}{2}x^2 \ln x\,dx + x^2 \int (-1)x\ln x\,dx + x^3 \int \frac{1}{2}\ln x\,dx$$

$$= \frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right)$$

$$y = y_h + y_p = c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right)$$