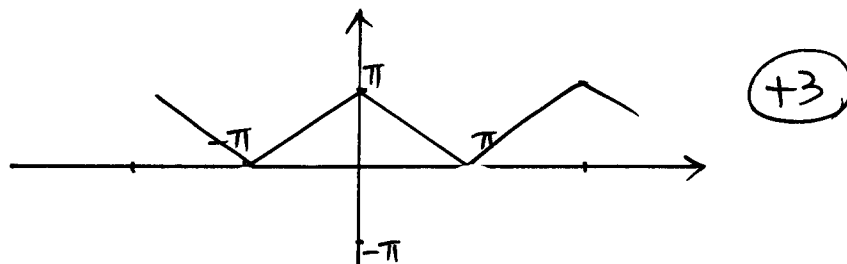


Final Exam: June 13, 2002

1. (20 points) Find the Fourier cosine series as well as the Fourier sine series of the following function. Sketch  $f(x)$  and its two periodic extensions.

$$f(x) = \pi - x, \quad 0 < x < \pi.$$

① Even Extension:

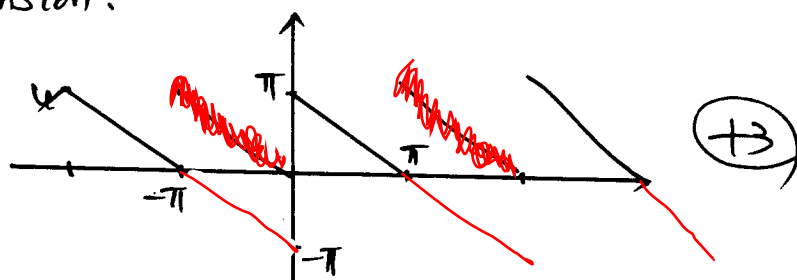


$$a_0 = \frac{1}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{1}{\pi} \left[ \pi x - \frac{1}{2} x^2 \right]_0^{\pi} = \frac{1}{2} \pi \quad (+2)$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx \\ &= \frac{2}{\pi} \left[ \frac{1}{n} (\pi - x) \sin nx \right]_0^{\pi} + \frac{2}{\pi} \cdot \frac{1}{n} \int_0^{\pi} \sin nx dx \quad (+3) \\ &= -\frac{2}{n\pi} \left[ \cos nx \right]_0^{\pi} = \frac{2}{n\pi} [1 - \cos n\pi] = \frac{2(1 - (-1)^n)}{n\pi} \end{aligned}$$

$$f(x) = \frac{1}{2} \pi + \frac{4}{\pi} \left[ \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right] \quad (+2)$$

② Odd Extension:



$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx \\ &= \frac{2}{\pi} \left[ -\frac{1}{n} (\pi - x) \cos nx \right]_0^{\pi} - \frac{2}{\pi} \cdot \frac{1}{n} \int_0^{\pi} \cos nx dx \quad (+4) \\ &= \frac{2}{\pi} - \frac{2}{n\pi} \left[ \frac{1}{n} \sin nx \right]_0^{\pi} = \frac{2}{n} \end{aligned}$$

$$f(x) = 2 \left[ \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right] \quad (+3)$$

2. (15 points) Show that the given integral represents the indicated function:

$$\int_0^{\infty} \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x}, \quad \text{if } x > 0.$$

Consider an even extension of  $f(x) = \frac{\pi}{2} e^{-x}$

$$\Rightarrow A(w) = \frac{2}{\pi} \int_0^{\infty} \frac{\pi}{2} e^{-v} \cos wv dv \quad \leftarrow (+5)$$

$$= \int_0^{\infty} e^{-v} \cos wv dv$$

$$= \left[ \frac{1}{w} e^{-v} \sin wv \right]_0^{\infty} + \frac{1}{w} \int_0^{\infty} e^{-v} \sin wv dv$$

$$= \frac{1}{w} \int_0^{\infty} e^{-v} \sin wv dv$$

$$= \frac{1}{w} \left[ -\frac{1}{w} e^{-v} \cos wv \right]_0^{\infty} - \frac{1}{w^2} \int_0^{\infty} e^{-v} \cos wv dv$$

$$= \frac{1}{w^2} - \frac{1}{w^2} A(w)$$

$$\therefore A(w) = \frac{1}{1+w^2} \quad \left. \vphantom{\frac{1}{1+w^2}} \right] (+5)$$

$$\Rightarrow \frac{\pi}{2} e^{-x} = \int_0^{\infty} A(w) \cos wx dw \quad \left. \vphantom{\int_0^{\infty}} \right] (+5)$$

$$= \int_0^{\infty} \frac{\cos wx}{1+w^2} dw$$

3. (15 points)

(a) (5 points) Compute  $\mathcal{F}_c\left(\frac{1}{1+x^2}\right)$  using the result of the previous problem.

(b) (10 points) Compute  $\mathcal{F}_s(xe^{-x^2/2})$  using the relation  $\mathcal{F}_c(e^{-x^2/2}) = e^{-w^2/2}$ .

$$(a) \mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cos wx}{1+x^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} e^{-w} \quad (+5)$$

$$= \sqrt{\frac{\pi}{2}} e^{-w}$$

$$(b) \mathcal{F}_s\left(xe^{-\frac{x^2}{2}}\right) = \mathcal{F}_s\left(-\left(e^{-\frac{x^2}{2}}\right)'\right) \quad (+4)$$

$$= -\mathcal{F}_s\left(\left(e^{-\frac{x^2}{2}}\right)'\right) \quad (+2)$$

$$= -\left(-w \mathcal{F}_c\left(e^{-\frac{x^2}{2}}\right)\right) \quad (+4)$$

$$= w \cdot e^{-\frac{w^2}{2}}$$