

Quiz #5 (CSE 400.001)
November 26, 2012 (Monday)

Name: _____ ID No: _____

1. (15 points) Solve the following problem from $x = 1$ to $x = 5$ with $h = 2.0$ using the Euler method:

$$x^2 y'' - 2xy' - 3y = 0, \quad y(1) = 3.0, \quad y'(1) = 6.0,$$

$$y'' - \frac{2}{x} y' - \frac{3}{x^2} y = 0$$
$$y_1 = y, \quad y_2 = y', \quad y_1(1) = 3.0, \quad y_2(1) = 6.0 \quad (+3)$$

$$\begin{cases} y_1' = y_2 \\ y_2' = \frac{2}{x} y_2 + \frac{3}{x^2} y_1 \end{cases}, \quad \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 3.0 \\ 6.0 \end{bmatrix} \quad (+3)$$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (+3)$$

$$\begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h \cdot \begin{bmatrix} y_{2,n} \\ \frac{2}{x_n} y_{2,n} + \frac{3}{x_n^2} y_{1,n} \end{bmatrix}$$

$$\begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} + 2.0 \begin{bmatrix} y_{2,0} \\ \frac{2}{1} * y_{2,0} + \frac{3}{1^2} * y_{1,0} \end{bmatrix} \quad (+3)$$
$$= \begin{bmatrix} 3.0 \\ 6.0 \end{bmatrix} + 2.0 \begin{bmatrix} 6.0 \\ 2 * 6.0 + 3 * 3.0 \end{bmatrix} = \begin{bmatrix} 15.0 \\ 48.0 \end{bmatrix}$$

$$\begin{bmatrix} y_{1,2} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} + 2.0 \begin{bmatrix} y_{2,1} \\ \frac{2}{3} * y_{2,1} + \frac{3}{3^2} * y_{1,1} \end{bmatrix} \quad (+3)$$
$$= \begin{bmatrix} 15.0 \\ 48.0 \end{bmatrix} + 2.0 \begin{bmatrix} 48.0 \\ \frac{2}{3} * 48.0 + \frac{1}{3} * 15.0 \end{bmatrix} = \begin{bmatrix} 111.0 \\ 122.0 \end{bmatrix}$$

2. (15 points) Consider a laterally insulated metal bar of length 1 and satisfying the heat equation $u_t = u_{xx}$. Suppose that the ends of the bar kept at temperature $u(0, t) = u(1, t) = 0$ and the initial temperature in the bar is $f(x) = 4x^2$, if $0 \leq x \leq 0.5$, and $f(x) = 4(x-1)^2$, if $0.5 \leq x \leq 1$. Applying the Crank-Nicolson method with $h = 0.2$ and $k = 0.08$, find the temperature $u(x, t)$ in the bar for $0 \leq t \leq 0.16$.

$$\frac{1}{k} (u_{\bar{i}, j+1} - u_{\bar{i}j}) = \frac{1}{2h^2} (u_{\bar{i+1}, j} - 2u_{\bar{i}j} + u_{\bar{i-1}, j}) + \frac{1}{2h^2} (u_{\bar{i+1}, j+1} - 2u_{\bar{i}, j+1} + u_{\bar{i-1}, j+1}) \quad (+2)$$

Let $r = \frac{k}{h^2} = \frac{0.08}{0.04} = 2$, then we have

$$u_{\bar{i}, j+1} - u_{\bar{i}j} = (u_{\bar{i+1}, j} - 2u_{\bar{i}j} + u_{\bar{i-1}, j}) + (u_{\bar{i+1}, j+1} - 2u_{\bar{i}, j+1} + u_{\bar{i-1}, j+1}) \quad (+3)$$

$$\underline{3u_{\bar{i}, j+1} - u_{\bar{i+1}, j+1} - u_{\bar{i-1}, j+1} = -u_{\bar{i}j} + u_{\bar{i+1}, j} + u_{\bar{i-1}, j}}$$

$$u_{10} = 0.16, \quad u_{20} = 0.64, \quad u_{30} = u_{20}, \quad u_{40} = u_{20} \quad (+2)$$

For $\bar{j}=0$:

$$(\bar{i}=1): \quad 3u_{11} - u_{21} - \overset{0}{u_{01}} = -u_{10} + u_{20} + \overset{0}{u_{00}} = 0.48 \quad (+2)$$

$$(\bar{i}=2): \quad -u_{11} + 3u_{21} - \underset{u_{31}}{u_{21}} = -u_{20} + \underset{u_{20}}{u_{30}} + u_{10} = 0.16$$

$$\Rightarrow \underline{u_{11} = 0.224, \quad u_{21} = 0.192} \quad (+2)$$

For $\bar{j}=1$:

$$(\bar{i}=1): \quad 3u_{12} - u_{22} - \overset{0}{u_{02}} = -u_{11} + u_{21} + \overset{0}{u_{01}} = -0.032$$

$$(\bar{i}=2): \quad -u_{12} + 2u_{22} = -u_{21} + \underset{u_{21}}{u_{31}} + u_{11} = 0.224 \quad (+2)$$

$$\Rightarrow \underline{u_{12} = 0.032, \quad u_{22} = 0.128} \quad (+2)$$