Engineering Mathematics I
(Comp 400.001)

Midterm Exam, October 29, 2014

< Solutions >

Problem	Score
1	
2	
3	
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Total	

Name: ____________________

ID No: ____________________

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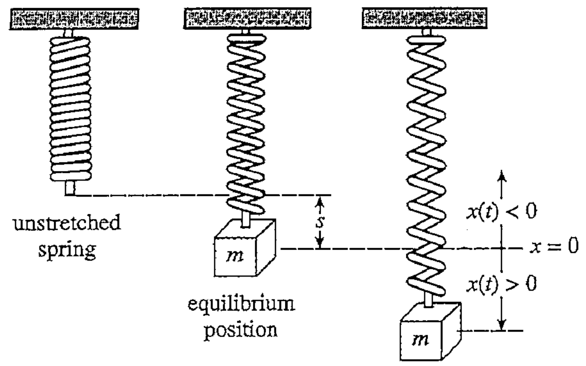
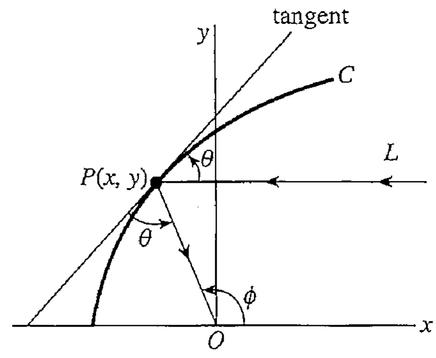
E-mail: ____________________

1. (20 points)



(A)



(B)

(a) (5 points) After a mass m is attached to a spring, the spring stretches s units and then hangs at rest in the equilibrium position as shown in Figure (A). After the spring/mass system has been set in motion, let $x(t)$ denote the directed distance of the mass beyond the equilibrium position. The restoring force of a spring is proportional to the elongation. Determine the differential equation for the displacement $x(t)$ at time t.

(b) (5 points) In Problem (a), determine a differential equation for the displacement $x(t)$ if the motion takes place in a medium that imparts a damping force on the spring/mass system that is proportional to the instantaneous velocity of the mass and acts in a direction opposite to that of motion.

(c) (10 points) As illustrated in Figure (B), light rays strike a plane curve C in such a manner that all rays L parallel to the x-axis are reflected to a single point O. Assuming that the angle of incidence is equal to the angle of reflection, determine a differential equation that describes the shape of the curve C. [Hint: Note that $\phi = 2\theta$.]

(a) $m x''(t) = -k x(t)$ for some $k > 0$

(b) $m x''(t) = -k x(t) - \alpha x'(t)$ for some $k, \alpha > 0$

(c) $\frac{y}{-x} = \tan(\pi - \varphi) = \tan(\pi - 2\theta) = -\tan 2\theta$

$y = x \cdot \tan 2\theta = x \cdot \frac{2\tan\theta}{1 - \tan^2\theta} = x \cdot \frac{2y'}{1-(y')^2}$

$\therefore \underline{y[1-(y')^2] = 2xy'}$

2. (10 points) Given a system of linear ODEs represented as a vector equation:

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{g}(t),$$

assume that $\mathbf{y}_1(t), \cdots, \mathbf{y}_n(t)$ are linearly independent solutions for the corresponding homogeneous system of equations: $\mathbf{y}'(t) = A\mathbf{y}(t)$. For the matrix $\mathbf{Y}(t) = [\mathbf{y}_1(t), \cdots, \mathbf{y}_n(t)]$, show that

$$\mathbf{Y}'(t) = A\mathbf{Y}(t).$$

$$\mathbf{Y}'(t) = [\mathbf{y}_1'(t), \cdots, \mathbf{y}_n'(t)] \quad (+2)$$

$$= [A\mathbf{y}_1(t), \cdots, A\mathbf{y}_n(t)] \quad (+3)$$

$$= A[\mathbf{y}_1(t), \cdots, \mathbf{y}_n(t)] \quad (+3)$$

$$= A\mathbf{Y}(t) \quad (+2)$$

3. (15 points) Solve the following differential equation

$$y'' - 2y' + 2y = e^x \tan x.$$

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0$$

$$y_h = Ae^x \cos x + Be^x \sin x \quad (+3)$$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x & e^x \cos x \end{vmatrix} = e^{2x} \quad (+3)$$

$$y_p = -e^x \cos x \int \frac{e^x \sin x}{e^{2x}} e^x \tan x \, dx + e^x \sin x \int \frac{e^x \cos x}{e^{2x}} e^x \tan x \, dx \quad (+3)$$

$$= -e^x \cos x \int \frac{1 - \cos^2 x}{\cos x} dx + e^x \sin x \int \sin x \, dx$$

$$= -e^x \cos x \int (\sec x - \cos x) dx + e^x \sin x [-\cos x]$$

$$= \underbrace{e^x \cos x \cdot \sin x}_{(+1)} - \underbrace{e^x \cos x \ln|\sec x + \tan x|}_{(+3)} \underbrace{- e^x \sin x \cos x}_{(+1)}$$

$$= -e^x \cos x \ln|\sec x + \tan x|$$

$$y = Ae^x \cos x + Be^x \sin x - e^x \cos x \ln|\sec x + \tan x| \quad (+1)$$

4. (15 points) Solve the following initial value problem:

$$\begin{aligned} y_1' &= 4y_1 - 2y_2 + 2u(t-1), \quad & y_1(0) = 0, \\ y_2' &= 3y_1 - y_2 + u(t-1), \quad & y_2(0) = 1/2. \end{aligned}$$

$$\left.\begin{aligned} sY_1 &= 4Y_1 - 2Y_2 + 2\cdot\frac{1}{s}e^{-s} \\ sY_2 - \frac{1}{2} &= 3Y_1 - Y_2 + \frac{1}{s}e^{-s} \end{aligned}\right] \quad (+2)$$

$$\left.\begin{aligned} (s-4)Y_1 + 2Y_2 &= \frac{2}{s}e^{-s} \\ -3Y_1 + (s+1)Y_2 &= \frac{1}{2} + \frac{1}{s}e^{-s} \end{aligned}\right] \quad (+1)$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{(s-1)(s-2)}\begin{bmatrix} s+1 & -2 \\ 3 & s-4 \end{bmatrix}\begin{bmatrix} \frac{2}{s}e^{-s} \\ \frac{1}{2} + \frac{1}{s}e^{-s} \end{bmatrix} \quad (+2)$$

$$= \begin{bmatrix} \dfrac{2e^{-s} - 1}{(s-1)(s-2)} \\ \dfrac{s+2}{s(s-1)(s-2)}e^{-s} + \dfrac{s-4}{2(s-1)(s-2)} \end{bmatrix} \quad (+2)$$

$$Y_1 = \left[\frac{1}{s-1} - \frac{1}{s-2} + \left(\frac{2}{s-2} - \frac{2}{s-1}\right)e^{-s}\right] \quad (+2)$$

$$Y_2 = \left[\frac{3/2}{s-1} - \frac{1}{s-2}\right] + \left[\frac{1}{s} - \frac{3}{s-1} + \frac{2}{s-2}\right]\cdot e^{-s} \quad (+2)$$

$$y_1(t) = e^{t} - e^{2t} + 2\left(e^{2(t-1)} - e^{t-1}\right)u(t-1) \quad (+2)$$

$$y_2(t) = \frac{3}{2}e^{t} - e^{2t} + \left(1 - 3e^{t-1} + 2e^{2(t-1)}\right)u(t-1) \quad (+2)$$

5. (15 points) Given a periodic function $f(t) = f(t+p)$, for $t > 0$, which is piecewise continuous and $|f(t)| \leq Me^{kt}$, for some k and M. Show that

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt \quad (+3)$$

(Let $x = t - p$)

$$= \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-s(x+p)} f(x+p)\,dx \quad (+3)$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-sx} \cdot e^{-sp} \cdot f(x)\,dx \quad (+3)$$

$$= \int_0^p e^{-st} f(t)\,dt + e^{-ps} \int_0^\infty e^{-sx} f(x)\,dx \quad (+3)$$

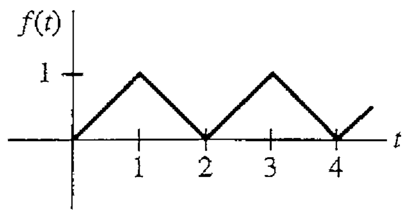
$$= \int_0^p e^{-st} f(t)\,dt + e^{-ps} F(s) \quad (+1)$$

$$\therefore F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt \quad (+2)$$

6. (15 points) Using the formula

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt,$$

find the Laplace transformation of the following periodic function:



$$f(t) = f(t+2) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 2-t & \text{if } 1 \le t \le 2. \end{cases}$$

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

(+2)

$$\int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}(2-t)\,dt$$

$$= \int_0^\infty e^{-st}\left[t\cdot(1-u(t-1))\right]dt$$

(+5)

$$+ \int_0^\infty e^{-st}\left[(2-t)\left(u(t-1)-u(t-2)\right)\right]dt$$

$$= \int_0^\infty e^{-st}\left[t + 2(1-t)u(t-1) + (t-2)u(t-2)\right]dt$$

(+5)

$$= \frac{1}{s^2} - 2\cdot\frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$= \frac{1}{s^2}(1-e^{-s})^2$$

(+2)

$$\therefore F(s) = \frac{1}{1-e^{-2s}}\cdot\frac{1}{s^2}(1-e^{-s})^2 = \frac{1-e^{-s}}{s^2(1+e^{-s})}$$

(+1)

7. (10 points) Solve the following integro-differential equation:

$$t - 2f(t) = \int_0^t (e^\tau - e^{-\tau}) f(t-\tau)\, d\tau \quad = (e^t - e^{-t}) * f(t)$$

$$\frac{1}{s^2} - 2F(s) = \left(\frac{1}{s-1} - \frac{1}{s+1}\right) F(s) \quad (+3)$$

$$\frac{1}{s^2} = \left(\frac{2}{s^2-1} + 2\right) F(s)$$

$$\frac{1}{s^2} = \frac{2s^2}{s^2-1} \cdot F(s) \quad (+3)$$

$$F(s) = \frac{s^2-1}{2s^4} = \frac{1}{2} \cdot \frac{1}{s^2} - \frac{1}{2} \cdot \frac{1}{s^4} \quad (+2)$$

$$f(t) = \frac{1}{2} t - \frac{1}{12} \cdot t^3 \quad (+2)$$