Unit Quaternions and 3D Rotations

Myung-Soo Kim
Seoul National University
http://cse.snu.ac.kr/mskim
http://3map.snu.ac.kr

Quaternions

- Sir William Hamilton discovered quaternions in 1843 as a generalization of complex numbers.
- Instead of one imaginary unit i, three imaginary units i, j, k are used in quaternions
- Each quaternion is represented as

$$q = w + xi + yj + zk$$

Quaternions

- Imaginary units

$$1 \cdot i = i, \quad 1 \cdot j = j, \quad 1 \cdot k = k,$$

$$i^2 = j^2 = k^2 = -1,$$

$$i \cdot j = k, \quad j \cdot i = -k,$$

$$j \cdot k = i, \quad k \cdot j = -i,$$

$$k \cdot i = j, \quad i \cdot k = -j.$$

Quaternions

- We may represent the quaternion as a 4–tuple of real numbers: $q = (w, x, y, z)$.
- Given two quaternions:

$$q_1 = (w_1, x_1, y_1, z_1),\ q_2 = (w_2, x_2, y_2, z_2),$$

$$\begin{aligned} q_1 + q_2 &= (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2), \\ q_1 \cdot q_2 &= (w_1 w_2 - \langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle, \\ &\quad w_1 (x_2, y_2, z_2) + w_2 (x_1, y_1, z_1) \\ &\quad + (x_1, y_1, z_1) \times (x_2, y_2, z_2)) \end{aligned}$$

Unit Quaternions

- Unit quaternions are closely related to 3D rotations. A unit quaternion can be represented as follows:

$$q = (w, x, y, z) = (\cos\theta, \sin\theta(a, b, c)),$$

where

$$\begin{aligned} (a, b, c) &= \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{w}\right). \end{aligned}$$

3D Rotations

- The unit quaternion

$$q = (\cos\theta, \sin\theta(a,b,c)) \in S^3$$

represents the rotation by angle 2θ about an axis $(a,b,c) \in S^2$.

- The rotation moves $(\alpha,\beta,\gamma) \in R^3$ to

$$(0,\bar{\alpha},\bar{\beta},\bar{\gamma}) = (\cos\theta, \sin\theta(a,b,c)) \cdot (0,\alpha,\beta,\gamma) \cdot (\cos\theta, -\sin\theta(a,b,c))$$

Rotation Matrix

$$
\begin{aligned}
& (w, x, y, z) \cdot (0, \alpha, \beta, \gamma) \cdot (w, -x, -y, -z) \\
= \;& (-(x\alpha + y\beta + z\gamma), w(\alpha, \beta, \gamma) + (x, y, z) \times (\alpha, \beta, \gamma)) \cdot (w, -x, -y, -z) \\
= \;& (-w(x\alpha + y\beta + z\gamma) + w(\alpha x + \beta y + \gamma z), \\
& \quad (x\alpha + y\beta + z\gamma)(x, y, z) + w^2(\alpha, \beta, \gamma) \\
& \quad +w(x, y, z) \times (\alpha, \beta, \gamma) - w(\alpha, \beta, \gamma) \times (x, y, z) \\
& \quad -((x, y, z) \times (\alpha, \beta, \gamma)) \times (x, y, z)) \\
= \;& (0, (x^2\alpha + xy\beta + xz\gamma, xy\alpha + y^2\beta + yz\gamma, xz\alpha + yz\beta + z^2\gamma) \\
& \quad (w^2\alpha, \qquad w^2\beta, \qquad w^2\gamma) \\
& \quad (2wy\gamma - 2wz\beta, \quad 2wz\alpha - 2wx\gamma, \quad 2wx\beta - 2wy\alpha) \\
& \quad (xy\beta + xz\gamma - z^2\alpha - y^2\alpha, \\
& \qquad\qquad xy\alpha + yz\gamma - x^2\beta - z^2\beta, \\
& \qquad\qquad\qquad xz\alpha + yz\beta - x^2\gamma - y^2\gamma))
\end{aligned}
$$

Rotation Matrix

$$(w, x, y, z) \cdot (0, \alpha, \beta, \gamma) \cdot (w, -x, -y, -z)$$

$$\begin{aligned}
= \quad & (0, (x^2\alpha + xy\beta + xz\gamma, xy\alpha + y^2\beta + yz\gamma, xz\alpha + yz\beta + z^2\gamma) \\
& \quad (w^2\alpha, \qquad w^2\beta, \qquad w^2\gamma) \\
& \quad (2wy\gamma - 2wz\beta, \quad 2wz\alpha - 2wx\gamma, \quad 2wx\beta - 2wy\alpha) \\
& \quad (xy\beta + xz\gamma - z^2\alpha - y^2\alpha, \\
& \qquad xy\alpha + yz\gamma - x^2\beta - z^2\beta, \\
& \qquad\qquad xz\alpha + yz\beta - x^2\gamma - y^2\gamma)) \\
= \quad & (0, (x^2 + w^2 - y^2 - z^2)\alpha + (2xy - 2wz)\beta + (2xz + 2wy)\gamma, \\
& \quad (2xy + 2wz)\alpha + (y^2 + w^2 - x^2 - z^2)\beta + (2yz - 2wx)\gamma, \\
& \quad (2xz - 2wy)\alpha + (2yz + 2wx)\beta + (w^2 + z^2 - x^2 - y^2)\gamma)
\end{aligned}$$

Rotation Matrix

$$
\begin{aligned}
& (w, x, y, z) \cdot (0, \alpha, \beta, \gamma) \cdot (w, -x, -y, -z) \\
= \; & (0, (x^2 + w^2 - y^2 - z^2)\alpha + (2xy - 2wz)\beta + (2xz + 2wy)\gamma, \\
& \quad (2xy + 2wz)\alpha + (y^2 + w^2 - x^2 - z^2)\beta + (2yz - 2wx)\gamma, \\
& \quad (2xz - 2wy)\alpha + (2yz + 2wx)\beta + (w^2 + z^2 - x^2 - y^2)\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}
&= \begin{bmatrix} x^2 + w^2 - y^2 - z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & y^2 + w^2 - x^2 - z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & w^2 + z^2 - x^2 - y^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}
\end{aligned}
$$

Rotation Matrix

- Each row/column is a unit vector.
- Rows/columns are mutually orthogonal each other.
- The determinant of rotation matrix is 1.
- Remark:

1. $R_{-q} = R_q$.

2. If $q_1, q_2 \in S^3$, then $q_2 \cdot q_1 \in S^3$.

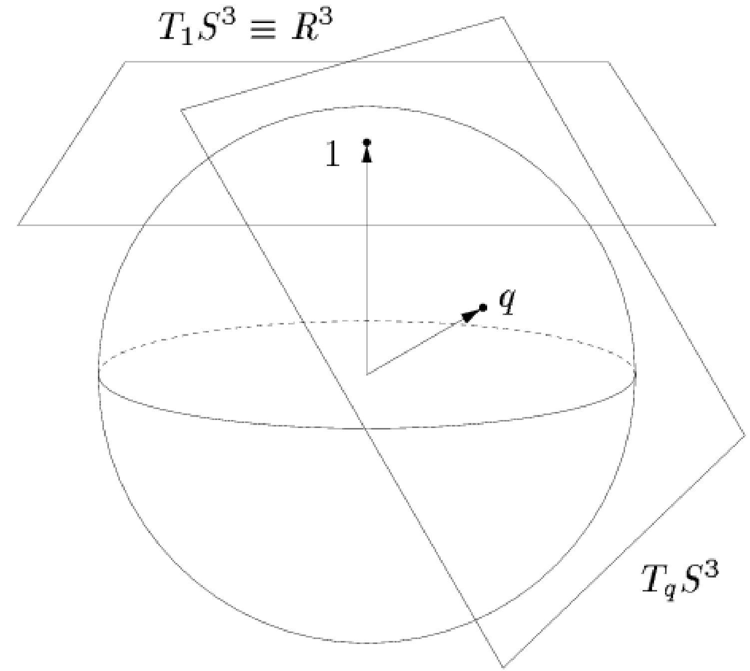
3. $R_{q_2} R_{q_1} = R_{q_2 \cdot q_1}$.

Quaternion Calculus

Given $q(t) \in S^3$,

$$q'(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$.



Angular Velocity

The rotated point $p(t) = R_{q(t)}(p)$ is in a sphere with radius $\|p\|$ and center $(0, 0, 0)$:

$$(0, p(t)) = q(t) \cdot (0, p) \cdot \overline{q(t)}.$$

Differentiating the above, we get

$$\begin{aligned} (0, p'(t)) &= (0, 2v(t)) \cdot (0, p(t)) \\ &= (0, 2v(t) \times p(t)), \end{aligned}$$

which means $\omega(t) = 2v(t)$.