Engineering Mathematics I (Comp 400.001)

Solution Set for Midterm Exam I : April 24, 2001

1. Find differential equations that describe the following curves.

   (a) (7 points) A curve in the xy-plane has the property that its slope at each point is equal to the sum of the square of the coordinates of the point.

$$y' = x^2 + y^2$$

   (b) (8 points) The graph of a nonnegative function has the property that the length of the arc between any two points on the graph is equal to the area of the region under the arc.

$$\int_a^x \sqrt{1 + y'(t)^2}dt = \int_a^x y(t)dt$$

2. (10 points) Find the general solution of the following differential equation:

$$\frac{y^2}{2} + 2ye^x + (y + e^x)\frac{dy}{dx} = 0$$

$$\underbrace{\left(\frac{y^2}{2}+2ye^x\right)}_{\substack{\parallel \\ P}}dx + \underbrace{(y+e^x)}_{\substack{\parallel \\ Q}}dy = 0 \qquad +2$$

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{y+e^x}\left(y + 2e^x - e^x\right) = 1 \qquad +2$$

$$F(x) = \exp\left(\int 1\, dx\right) = e^x \qquad +2$$

$$\left(\frac{1}{2}y^2e^x + 2ye^x\right)dx + \left(ye^x + e^{2x}\right)dy = 0 \quad : \text{exact}$$

$$u(x,y) = \frac{1}{2}y^2e^x + ye^{2x} + k(y) \qquad +2$$

$$\frac{\partial u}{\partial y} = ye^x + e^{2x} + k'(y) = ye^x + e^{2x}$$

$$\therefore k(y) = \text{const}$$

$$\therefore u(x,y) = \frac{1}{2}y^2e^x + ye^{2x} + c = 0 \qquad +2$$

3. (10 points) Using a particular solution $y = e^{-x}$, find the general solution of the following differential equation:

$$xy'' + (x-1)y' - y = 0,\ x > 0.$$

$y'' + \underbrace{(1-\frac{1}{x})}_{P(x)} y' \underbrace{- \frac{1}{x}}_{q(x)} y = 0$, $x > 0$ (+2)

$u = \frac{1}{y_1^2} \cdot e^{-\int p(x)dx} = e^{2x} \cdot e^{-(x-\ln x)} = e^{x} \cdot x = xe^{x}$ (+4)

$y_2 = y_1 \int u(x)dx = e^{-x} \int xe^{x} dx = e^{-x}[xe^{x} - e^{x}] = (x-1)$ (+4)

$y = k_1 e^{-x} + k_2 (x-1)$

4. (10 points) Find the general solution of the following equation:

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}.$$

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3$

$y_h = c_1 e^{3x} + c_2 x e^{3x}$ (+2)

$y_p = a + bx + cx^2 + dx^2 e^{3x}$

$y_p' = b + 2cx + 2dx e^{3x} + 3dx^2 e^{3x}$

$y_p'' = 2c + 2de^{3x} + 12dxe^{3x} + 9dx^2 e^{3x}$ (+3)

$y_p'' - 6y_p' + 9y_p = 9a - 6b + 2c + (9b - 12c)x + 9cx^2 + 2de^{3x}$

$$\left.\begin{aligned} 9a - 6b + 2c &= 2 \\ 9b - 12c &= 0 \\ 9c &= 6 \\ 2d &= -12 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= \frac{2}{3} \\ b &= \frac{8}{9} \\ c &= \frac{2}{3} \\ d &= -6 \end{aligned}$$

$\therefore y_p = \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2 e^{3x}$ (+5)

$\therefore y = y_h + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2 e^{3x}$

5. (15 points) Find the general solution of the following equation:

$$x^2y'' - 3xy' + 3y = 2x^4e^x.$$

$m(m-1) - 3m + 3 = 0,$

$m = 1, 3$

$y_1 = x, \quad y_2 = x^3$ (+3)

$y_h = c_1x + c_2x^3$

$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$ (+3)

$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \underbrace{2x^2e^x}_{r(x)}$

$y_p = -y_1\int \frac{y_2 \cdot r(x)}{W(x)}dx + y_2 \int \frac{y_1 \cdot r(x)}{W(x)}dx$ (+3)

$= -x\int \frac{x^3 \cdot 2x^2e^x}{2x^3}dx + x^3\int \frac{x \cdot 2x^2e^x}{2x^3}dx$

$= -x\int x^2e^x dx + x^3\int e^x dx$

$= -x\left(\cancel{x^2e^x} - \int 2xe^x dx\right) + \cancel{x^3e^x}$

$= x\left(2xe^x - \int 2e^x dx\right)$

$= 2x^2e^x - 2xe^x.$ (+4)

$\therefore y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$ (+2)

6. (10 points) Find the Laplace transform of the following function:

$$te^{2t}f'(t)$$

$$\begin{aligned}
\mathcal{L}[f(t)] &= F(s) \\
\mathcal{L}[f'(t)] &= sF(s) - f(0) \qquad +3 \\
\mathcal{L}[e^{2t}f'(t)] &= (s-2)F(s-2) - f(0) \qquad +3 \\
\mathcal{L}[t(e^{2t}f'(t))] &= -\frac{d}{ds}[(s-2)F(s-2) - f(0)] \\
&= -F(s-2) + (2-s)F'(s-2) \qquad +4
\end{aligned}$$

7. (15 points) Find the Laplace transform of the following function:

$$t^2\Big(\int_0^t \tau \sin \tau d\tau\Big)$$

$$\begin{aligned}
\mathcal{L}[\sin t] &= \frac{1}{s^2+1} \qquad +2 \\
\mathcal{L}[t\sin t] &= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2} \qquad +2 \\
\mathcal{L}[\int_0^t \tau \sin \tau d\tau] &= \frac{1}{s}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2} \qquad +4 \\
\mathcal{L}[t^2\left(\int_0^t \tau \sin \tau d\tau\right)] &= \frac{d^2}{ds^2}\left[\frac{2}{(s^2+1)^2}\right] \qquad +4 \\
&= \frac{d}{ds}\left[\frac{-4(2s)}{(s^2+1)^3}\right] = \frac{-8(s^2+1)^3 + 8s\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6} \\
&= \frac{-8s^2 - 8 + 48s^2}{(s^2+1)^4} = \frac{40s^2-8}{(s^2+1)^4} \qquad +3
\end{aligned}$$

8. (15 points) Find f and g satisfying the following simultaneous equations:

$$f(t) + \int_0^t (t-\tau)g(\tau)d\tau = \sin 2t$$

$$g(t) + \int_0^t (t-\tau)f(\tau)d\tau = 0$$

+5
$$\left\{\begin{aligned} F(s) + \tfrac{1}{s^2}G(s) &= \tfrac{2}{s^2+4} \\ \tfrac{1}{s^2}F(s) + G(s) &= 0 \end{aligned}\right. \Longrightarrow \left\{\begin{aligned} s^2F(s) + G(s) &= \tfrac{2s^2}{s^2+4} \\ F(s) + s^2G(s) &= 0 \end{aligned}\right.$$

$$\begin{bmatrix} F(s) \\ G(s) \end{bmatrix} = \frac{1}{s^4-1}\begin{bmatrix} s^2 & -1 \\ -1 & s^2 \end{bmatrix}\begin{bmatrix} \frac{2s^3}{s^2+4} \\ 0 \end{bmatrix} \qquad +3$$

$$\begin{aligned}
F(s) &= \frac{2s^4}{(s^4-1)(s^2+4)} = \frac{1}{10}\cdot\frac{1}{s-1} + \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{16}{15}\cdot\frac{2}{s^2+4} \\
G(s) &= \frac{-2s^2}{(s^4-1)(s^2+4)} = -\frac{1}{10}\cdot\frac{1}{s-1} - \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{4}{15}\cdot\frac{2}{s^2+4} \qquad +4
\end{aligned}$$

$$\begin{aligned}
f(t) &= \frac{1}{5}\cosh t - \frac{1}{3}\sin t + \frac{16}{15}\sin 2t \\
g(t) &= -\frac{1}{5}\cosh t - \frac{1}{3}\sin t + \frac{4}{15}\sin 2t \qquad +3
\end{aligned}$$