Engineering Mathematics I
(Comp 400.001)

Midterm Exam I: April 8, 2003

1. (15 points) In each of the following problems, formulate a differential equation that models the given problem. You don't need to solve the differential equations thus formulated.

   (a) (5 points) Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation governing the number of people $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between the number of students with the flu and the number of students who have not yet been exposed to it.

   (b) (5 points) For high-speed motion through the air – such as the skydiver falling before the parachute is opened – air resistance is closer to a power of the instantaneous velocity. Determine a differential equation for the velocity $v(t)$ of a falling body of mass m if air resistance is proportional to the square of the instantaneous velocity.

   (c) (5 points) In the theory of learning, the rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. Suppose M denotes the total amount of a subject to be memorized and $A(t)$ is the amount memorized at time t. Moreover, assume that the rate at which material is forgotten is proportional to the amount memorized in time t. Determine a differential equation for the amount $A(t)$.

ⓐ $\underbrace{x'(t)}_{+1} = \overset{+1}{k}\,\underbrace{x(1000-x)}_{+3}$

ⓑ $\underbrace{mv'(t)}_{+1} = \overbrace{mg - \underbrace{kv^2}_{+2}}^{+2}$

ⓒ $\underbrace{A'(t)}_{+1} = \underbrace{k_1(M-A)}_{+2} - \underbrace{k_2A}_{+2}$

2. (10 points) Find an explicit solution for the following initial value problem

$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy}, \quad y(1) = -\sqrt{2}.$$

$$u = \frac{y}{x} \Rightarrow y = ux, \quad y' = u'x + u \qquad +2$$

$$\left.\begin{aligned} u'x + \cancel{u} = \cancel{u} - \frac{1}{u}, \quad & u\,du = -\frac{1}{x}\,dx \\ & \frac{1}{2}u^2 = -\ln|x| + C \end{aligned}\right] +3$$

$$y^2 = 2Cx^2 - 2x^2 \ln|x|, \quad 2 = 2C, \quad C = 1 \qquad +2$$

$$y = \underbrace{-\sqrt{2}\,|x| \cdot \sqrt{1 - \ln|x|}}_{+2}, \quad \text{for } \underbrace{-e < x < e,\ x \neq 0}_{+1}$$

or

$$y = \underbrace{-\sqrt{2}\,x\sqrt{1 - \ln x}}_{+2}, \quad \text{for } \underbrace{0 < x < e}_{+1}$$

3. (15 points) Solve the following differential equation

$$y'' - 2y' + 2y = e^x \tan x.$$

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0, \quad y_h = Ae^x \cos x + Be^x \sin x \qquad +3$$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x & e^x \cos x \end{vmatrix} = e^{2x}$$

$$+3$$

$$\left.\begin{aligned} y_p(x) = & -e^x \cos x \int \frac{e^x \sin x}{e^{2x}} e^x \tan x\,dx \\ & + e^x \sin x \int \frac{e^x \cos x}{e^{2x}} e^x \tan x\,dx \end{aligned}\right] +3$$

$$= -e^x \cos x \int \frac{1 - \cos^2 x}{\cos x}\,dx + e^x \sin x \int \sin x\,dx$$

$$= -e^x \cos x \int (\sec x - \cos x)\,dx + e^x \sin x\,[-\cos x]$$

$$= \underbrace{e^x \cos x \cdot \sin x}_{+1} \underbrace{- e^x \cos x \ln|\sec x + \tan x|}_{+3} \underbrace{- e^x \sin x \cos x}_{+1}$$

$$= -e^x \cos x \ln|\sec x + \tan x|$$

$$y = \underbrace{Ae^x \cos x + Be^x \sin x - e^x \cos x \ln|\sec x + \tan x|}_{+1}$$

4. (20 points) Solve the following initial value problem

$$y'' + 5y' + 6y = u(t-1) + \delta(t-2), \quad y(0) = 0, \; y'(0) = 1.$$

$$s^2Y - 1 + 5sY + 6Y = \frac{1}{s}e^{-s} + e^{-2s}$$
$$(s+2)(s+3)Y = \frac{1}{s}e^{-s} + e^{-2s} + 1$$ (+3)

$$Y = \frac{1}{(s+2)(s+3)} + \frac{1}{s(s+2)(s+3)}e^{-s} + \frac{1}{(s+2)(s+3)}e^{-2s}$$ (+3)

$$\frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$
$$\frac{1}{s(s+2)(s+3)} = \frac{1}{s}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$
$$= \frac{1/6}{s} - \frac{1/2}{s+2} + \frac{1/3}{s+3}$$ (+3)

$$y(t) = \underbrace{e^{-2t} - e^{-3t}}_{(+3)} + \underbrace{\left[\frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)}\right]u(t-1)}_{(+4)}$$
$$+ \underbrace{\left[e^{-2(t-2)} - e^{-3(t-2)}\right]u(t-2)}_{(+4)}$$

or

$$= \begin{cases} e^{-2t} - e^{-3t} & \text{if } 0 < t < 1 \quad (+3) \\ \frac{1}{6} + (1 - \frac{1}{2}e^{2})e^{-2t} + (\frac{1}{3}e^{3} - 1)e^{-3t} & \text{if } 1 < t < 2 \quad (+4) \\ \frac{1}{6} + (1 - \frac{1}{2}e^{2} + e^{4})e^{-2t} + (\frac{1}{3}e^{3} - 1 - e^{6})e^{-3t} & \text{if } t > 2 \quad (+4) \end{cases}$$

5. (10 points) Solve the following integral equation

$$y(t) = 1 + \int_0^t y(\tau)d\tau.$$

$$Y = \frac{1}{s} + \frac{1}{s}Y$$ (+7)

$$\frac{s-1}{s}Y = \frac{1}{s}$$
$$Y = \frac{1}{s-1}$$ (+2)

$$y(t) = e^{t}$$ (+1)

6. (15 points) Solve the following initial value problem

$$y' - y = te^t \sin t, \quad y(0) = 0.$$

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$, $\mathcal{L}[e^t \sin t] = \frac{1}{(s-1)^2+1}$

$\mathcal{L}[te^t \sin t] = -\frac{d}{ds}\left[\frac{1}{(s-1)^2+1}\right]$

$= -\frac{-2(s-1)}{((s-1)^2+1)^2} = \frac{2(s-1)}{((s-1)^2+1)^2}$ (+4)

$sY - Y = \frac{2(s-1)}{((s-1)^2+1)^2}$

$Y = \frac{2}{((s-1)^2+1)^2} = 2 \cdot \frac{1}{(s-1)^2+1} \cdot \frac{1}{(s-1)^2+1}$ (+4)

$y(t) = 2e^t \sin t * \sin t$

$= 2e^t \int_0^t \sin\tau \cdot \sin(t-\tau)\,d\tau$ (+4)

$= 2e^t \cdot \frac{1}{2}\left[\int_0^t \cos(2\tau - t)\,d\tau - \int_0^t \cos t\,d\tau\right]$

$= e^t\left[\frac{1}{2}\sin(2\tau - t)\Big|_0^t - t\cos t\right]$ (+3)

$= e^t \sin t - te^t \cos t$

7. (15 points) Find the inverse Laplace transform of the following function

$$\ln \frac{(s^2+\pi^2)^2}{(s^2+6s+10)^2}.$$

$F(s) = 2\ln\frac{(s^2+\pi^2)}{(s^2+6s+10)}$

$F'(s) = 2 \cdot \frac{\frac{2s(s^2+6s+10) - (s^2+\pi^2)(2s+6)}{(s^2+6s+10)^2}}{\frac{(s^2+\pi^2)}{(s^2+6s+10)}}$

$= 2 \cdot \frac{2s(s^2+6s+10) - 2(s^2+\pi^2)(s+3)}{(s^2+\pi^2)(s^2+6s+10)}$

$= 4 \cdot \left[\frac{s}{s^2+\pi^2} - \frac{s+3}{(s+3)^2+1}\right]$ (+7)

$-tf(t) = 4 \cdot \left[\cos \pi t - e^{-3t}\cos t\right]$ (+7)

$f(t) = \frac{4}{t}\left[e^{-3t}\cos t - \cos \pi t\right]$ (+1)