Engineering Mathematics I
(Comp 400.001)

Midterm Exam II: May 15, 2003

1. (10 points) Consider the iteration defined by

$$x_{n+1} = x_n - f(x_n)\left[\frac{f(x_n)}{f(x_n + f(x_n)) - f(x_n)}\right].$$

This is known as Steffenson's method; and it has local quadratic convergence. Starting from $x_0 = 2$, apply two steps of Steffenson's method to

$$f(x) = x^2 - 2,$$

for the approximation of $\sqrt{2}$.
(**Extra Credit: 30 points**) Show the local quadratic convergence of the above iteration method.

$x_0 = 2$

$$x_1 = 2 - f(2)\cdot\left[\frac{f(2)}{f(2+f(2)) - f(2)}\right]$$

$$= 2 - 2\cdot\frac{2}{f(4)-2} = 2 - \frac{4}{12} = \frac{5}{3} \approx 1.67$$

+4

$$x_2 = \frac{5}{3} - f\left(\frac{5}{3}\right)\cdot\left[\frac{f\left(\frac{5}{3}\right)}{f\left(\frac{5}{3}+f\left(\frac{5}{3}\right)\right) - f\left(\frac{5}{3}\right)}\right]$$

$$= \frac{5}{3} - \frac{7}{9}\cdot\frac{7/9}{f\left(\frac{5}{3}+\frac{7}{9}\right) - \frac{7}{9}}$$

$$= \frac{5}{3} - \frac{7}{37} = \frac{164}{111} \approx 1.48$$

+6

계산하지 않으면 −2

Extra Credit

$$g(x) = x - f(x) \cdot \left[\frac{f(x)}{f(x+f(x)) - f(x)} \right]$$

$$g'(x) = 1 - f'(x) \cdot \left[\frac{f(x)}{f(x+f(x)) - f(x)} \right]$$

$$- f(x) \cdot \frac{f'(x)\left[f(x+f(x)) - f(x)\right] - f(x)\left[f'(x+f(x))(1+f'(x)) - f'(x)\right]}{\left(f(x+f(x)) - f(x)\right)^2}$$

$$= 1 - 2 f'(x) \cdot \frac{f(x)}{f(x+f(x)) - f(x)}$$

$$+ f(x)^2 \cdot \frac{f'(x+f(x))(1+f'(x)) - f'(x)}{\left[f(x+f(x)) - f(x)\right]^2}$$

$$g'(s) = \lim_{x \to s} g'(x)$$

$$= 1 - 2 f'(s) \cdot \lim_{x \to s} \frac{1}{\dfrac{f(x+f(x)) - f(x)}{x + f(x) - x}}$$

$$+ \left[f'(s)(1+f'(s)) - f'(s) \right] \cdot \lim_{x \to s} \frac{1}{\left[\dfrac{f(x+f(x)) - f(x)}{x + f(x) - x} \right]^2}$$

$$= 1 - 2 f'(s) \cdot \frac{1}{f'(s)} + f'(s)^2 \cdot \frac{1}{f'(s)^2} = 0$$

Hence, Steffenson's method has local quadratic convergence.

2. (20 points) Show that, when the matrix A is accurate, an inaccuracy $\delta\mathbf{b}$ of the right side $\mathbf{b}$ causes an inaccuracy $\delta\mathbf{x}$ satisfying

$$\frac{||\delta\mathbf{x}||}{||\mathbf{x}||} \le \kappa(A)\frac{||\delta\mathbf{b}||}{||\mathbf{b}||}$$

$$A(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}+\delta\mathbf{b}$$

$$A\mathbf{x} + A\delta\mathbf{x} = \mathbf{b}+\delta\mathbf{b}$$

$$A\delta\mathbf{x} = \delta\mathbf{b}$$

$$\delta\mathbf{x} = A^{-1}\delta\mathbf{b}$$

+5

$$\frac{||\delta\mathbf{x}||}{||\mathbf{x}||} = \frac{||A^{-1}\delta\mathbf{b}||}{||\mathbf{x}||}$$

$$\le \frac{||A^{-1}||\,||\delta\mathbf{b}||}{||\mathbf{x}||}$$

+5

$$= \frac{||A\mathbf{x}||\cdot||A^{-1}||\,||\delta\mathbf{b}||}{||\mathbf{b}||\cdot||\mathbf{x}||}$$

+5

$$\le \frac{||A||\,||\mathbf{x}||\cdot||A^{-1}||\,||\delta\mathbf{b}||}{||\mathbf{b}||\cdot||\mathbf{x}||}$$

$$= ||A||\cdot||A^{-1}||\cdot\frac{||\delta\mathbf{b}||}{||\mathbf{b}||}$$

$$= \kappa(A)\cdot\frac{||\delta\mathbf{b}||}{||\mathbf{b}||}$$

+5

3. (15 points) Table 1 shows the result of applying the Runge-Kutta method to the following initial value problem with $h = 0.2$:

$$y' = -(y+1)(y+3), \quad \text{for } 0 \le x \le 1, \quad y(0) = -2.$$

Fill in the blank; and show your work for partial credit.

x_i	y_i
0.2	−1.80263
0.4	
0.6	−1.46296
0.8	−1.33598
1.0	−1.23843

Table 1: Runge-Kutta Method

$x_1 = 0.2, \quad y_1 = -1.80263$

$h = 0.2, \quad f(x,y) = -(y+1)(y+3)$ +2

$k_1 = 0.2 * f(x_1, y_1) = 0.19221$ +2

$k_2 = 0.2 * f(x_1 + 0.1, y_1 + 0.5k_1) = 0.18277$ +3

$k_3 = 0.2 * f(x_1 + 0.1, y_1 + 0.5k_2) = 0.18332$ +3

$k_4 = 0.2 * f(x_2, y_1 + k_3) = 0.17101$ +3

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -1.62006$ +2

4. (25 points) Using $h = 1/3$ and $k = 1/2$, approximate the solution to the following elliptic equation

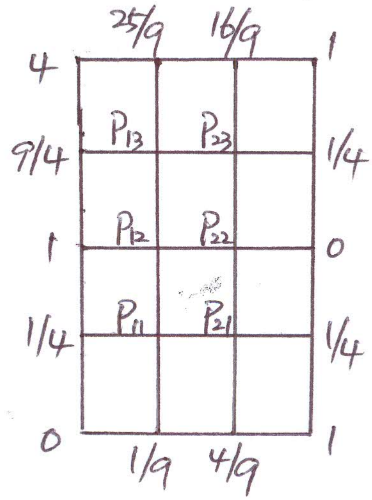
$$u_{xx} + 9u_{yy} = 4, \quad 0 < x < 1,\ 0 < y < 2$$

with boundary conditions:

$$u(x,0) = x^2, \quad u(x,2) = (x-2)^2, \quad 0 \le x \le 1;$$
$$u(0,y) = y^2, \quad u(1,y) = (y-1)^2, \quad 0 \le y \le 2.$$

Set up a system of linear equations.

+5



i	j	x_i	y_j	$u(x_i, y_j)$	i	j	x_i	y_j	$u(x_i, y_j)$
1	1	1/3	1/2		2	1	2/3	1/2	
1	2	1/3	1		2	2	2/3	1	
1	3	1/3	3/2		2	3	2/3	3/2	

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + 9\cdot\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 4 \qquad +5$$

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + 4(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = 4/9$$
$$u_{i+1,j} - 10u_{i,j} + u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} = 4/9 \qquad +5$$

$$\begin{cases} P_{11}: & -10u_{11} + u_{21} + 4u_{12} = -1/4 \\ P_{21}: & u_{11} - 10u_{21} + 4u_{22} = -57/36 \\ P_{12}: & 4u_{11} - 10u_{12} + u_{22} + 4u_{13} = -5/9 \\ P_{22}: & 4u_{21} + u_{12} - 10u_{22} + 4u_{23} = 4/9 \\ P_{13}: & 4u_{12} - 10u_{13} + u_{23} = -155/12 \\ P_{23}: & 4u_{22} + u_{13} - 10u_{23} = -83/12 \end{cases} \qquad +10$$

↳ 하나당 +2 점씩

5. (30 points) Consider the following parabolic equation (**warning: this is different from $u_t = u_{xx}$!**):

$$u_t = 4u_{xx} + 100, \quad 0 \le x \le 1,\ 0 \le t \le 0.01,$$

with boundary conditions

$$\begin{cases} u(0,t) = 0, & \text{for } 0 \le t \le 0.01 \\ u(1,t) = 0, & \text{for } 0 \le t \le 0.01 \\ u(x,0) = \begin{cases} x, & \text{if } 0 \le x \le 0.2 \\ 0.25(1-x), & \text{if } 0.2 \le x \le 1 \end{cases} \end{cases}$$

Approximate the solution to above equation using Crank-Nicolson method with $h = 0.2$ and $k = 0.01$ for $0 \le t \le 0.01$. Set up the Gauss-Seidel Iteration that solves this problem.

$$\frac{u_{i,j+1} - u_{i,j}}{k} = 2\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right] + 2\left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right] + 100 \quad +8$$

$$2u_{i,j+1} - \cancel{2u_{i,j}} = u_{i+1,j} - \cancel{2u_{i,j}} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + 2 \quad +4$$

$$4u_{i,j+1} - u_{i+1,j+1} - u_{i-1,j+1} = u_{i+1,j} + u_{i-1,j} + 2 \quad +4$$

$$(u_{10} = 0.2,\ u_{20} = 0.15,\ u_{30} = 0.10,\ u_{40} = 0.05) \quad +4$$

$j=0$:

$$\begin{aligned} (i=1)&: \ 4u_{11} - u_{21} = u_{20} + u_{00} + 2 = 2.15 \\ (i=2)&: \ -u_{11} + 4u_{21} - u_{31} = u_{30} + u_{10} + 2 = 2.3 \\ (i=3)&: \ -u_{21} + 4u_{31} - u_{41} = u_{40} + u_{20} + 2 = 2.2 \\ (i=4)&: \ -u_{31} + 4u_{41} = u_{30} + 2 = 2.1 \end{aligned} \quad +8$$

$$\begin{cases} u_{11} = 0.25\,u_{21} + 0.5375 \\ u_{21} = 0.25(u_{11} + u_{31}) + 0.575 \\ u_{31} = 0.25(u_{21} + u_{41}) + 0.55 \\ u_{41} = 0.25\,u_{31} + 0.525 \end{cases} \quad +2$$