# Engineering Mathematics I (Comp 400.001)

## Midterm Exam II: November 17, 2004

< Solution Set >

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Name: ____________________

ID No: ____________________

Dept: ____________________

E-mail: ____________________

1. (15 points) Given a square matrix $A = [a_{ij}]$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, show that

(a) (5 points) the determinant of $A = \lambda_1 \lambda_2 \cdots \lambda_n$

(b) (10 points) the trace of $A$ $(= \sum_{i=1}^{n} a_{ii}) = \sum_{k=1}^{n} \lambda_k$

(a) $f(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ +3

$$\left.\begin{aligned} \det(A) = f(0) &= (-1)^n (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) \\ &= (-1)^n (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \\ &= \lambda_1 \lambda_2 \cdots \lambda_n \end{aligned}\right] \text{+2}$$

(b)

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} - \lambda \end{bmatrix}$$

$$f(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \quad \text{+2}$$

$$= \underline{(-1)^n \lambda^n + (-1)^{n+1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \lambda^{n-1}} + \cdots + (\lambda_1 \lambda_2 \cdots \lambda_n)$$

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) + \left[\begin{matrix}\text{polynomial in } \lambda \\ \text{of degree at most } (n-2)\end{matrix}\right]$$

+5

$$= \underline{(-1)^n \lambda^n + (-1)^{n-1} (a_{11} + a_{22} + \cdots + a_{nn}) \lambda^{n-1}} + \cdots + \det(A)$$

+2

$$\therefore \text{ trace of } A = \sum_{i=1}^{n} a_{ii} = \sum_{k=1}^{n} \lambda_k$$

+1

2. (20 points) We apply the improved Euler method to the following initial value problem with $h = 0.2$:

$$y'' = -y + 2x, \quad \text{for } 0 \le x \le 1, \quad y(0) = -1, \; y'(0) = 8.$$

Derive the recursive formulas for $y_{1,n+1}$ and $y_{2,n+1}$ in terms of $x_n, y_{1,n}$, and $y_{2,n}$.

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 + 2x \end{cases} \quad +2 \qquad \Rightarrow \qquad \begin{cases} f_1(x, y_1, y_2) = y_2 \\ f_2(x, y_1, y_2) = -y_1 + 2x \end{cases} \quad +3$$

$$\mathbb{K}_1 = 0.2 \begin{bmatrix} f_1(x_n, y_{1,n}, y_{2,n}) \\ f_2(x_n, y_{1,n}, y_{2,n}) \end{bmatrix} = \begin{bmatrix} 0.2\, y_{2,n} \\ -0.2\, y_{1,n} + 0.4 x_n \end{bmatrix} \quad +4$$

$$\mathbb{K}_2 = 0.2 \begin{bmatrix} f_1(x_n + 0.2,\; y_{1,n} + 0.2 y_{2,n},\; y_{2,n} - 0.2 y_{1,n} + 0.4 x_n) \\ f_2(\text{—}''\text{—},\; \text{—}''\text{—},\; \text{—}''\text{—}) \end{bmatrix} \quad +4$$

$$= 0.2 \begin{bmatrix} y_{2,n} - 0.2 y_{1,n} + 0.4 x_n \\ -y_{1,n} - 0.2 y_{2,n} + 2x_n + 0.4 \end{bmatrix} \quad +4$$

$$= \begin{bmatrix} 0.2\, y_{2,n} - 0.04\, y_{1,n} + 0.08\, x_n \\ -0.2\, y_{1,n} - 0.04\, y_{2,n} + 0.4 x_n + 0.08 \end{bmatrix}$$

$$\begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + \frac{1}{2}(\mathbb{K}_1 + \mathbb{K}_2)$$

$$= \begin{bmatrix} y_{1,n} + 0.2 y_{2,n} - 0.02 y_{1,n} + 0.04 x_n \\ y_{2,n} - 0.2 y_{1,n} - 0.02 y_{2,n} + 0.4 x_n + 0.04 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98\, y_{1,n} + 0.2\, y_{2,n} + 0.04\, x_n \\ -0.2\, y_{1,n} + 0.98\, y_{2,n} + 0.4 x_n + 0.04 \end{bmatrix} \quad +3$$

3. (30 points) Consider the following heat equation

$$u_t = u_{xx}, \quad 0 \le x \le 1,\ 0 \le t \le 0.02,$$

with boundary conditions

$$\begin{cases} u_x(0,t) = t, & \text{for } 0 \le t \le 0.02 \\ u_x(1,t) = t^2, & \text{for } 0 \le t \le 0.02 \\ u(x,0) = 0.5 - |x - 0.5| \end{cases}$$

Approximate the solution to the above equation using the Crank-Nicolson method with $h = 0.2$ and $k = 0.02$ for $0 \le t \le 0.02$. Set up the Gauss-Seidel Iteration that solves this problem.

$$\frac{u_{i,j+1}-u_{i,j}}{k} = \frac{1}{2}\left[\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right] + \frac{1}{2}\left[\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right] \quad (+2)$$

$$\cancel{4}\,6\,u_{i,j+1} - \cancel{4}\,2\,u_{i,j} = u_{i+1,j} - \cancel{2u_{i,j}} + u_{i-1,j} + u_{i+1,j+1} - \cancel{2u_{i,j+1}} + u_{i-1,j+1} \quad (+2)$$

$$6u_{i,j+1} - u_{i+1,j+1} - u_{i-1,j+1} = u_{i+1,j} + 2u_{i,j} + u_{i-1,j} \quad (+2)$$

$$(u_{00}=0,\ u_{1,0}=0.2,\ u_{2,0}=0.4,\ u_{3,0}=0.4,\ u_{4,0}=0.2,\ u_{5,0}=0) \quad (+1)$$

① $u_{1,j} - u_{-1,j} = 2h\,(0.02j) = 0.008j$ (+2)

$$6u_{0,1} - u_{1,1} - u_{-1,1} = u_{-1,0} + 2u_{0,0} + u_{1,0} = 2u_{0,0} + 2u_{1,0} = 0.4 \quad (+2)$$

$$6u_{0,1} - 2u_{1,1} + 0.008 = 0.4 \quad (+4)$$

$$u_{0,1} = \frac{1}{3}u_{1,1} + \frac{49}{75} \quad (+1)$$

② $u_{6,j} - u_{4,j} = 2h\,(0.02j)^2 = 0.00016j^2$ (+2)

$$6u_{5,1} - u_{4,1} - u_{6,1} = u_{4,0} + 2u_{5,0} + u_{6,0} = 2u_{4,0} + 2u_{5,0} = 0.4 \quad (+2)$$

$$6u_{5,1} - 2u_{4,1} - 0.00016 = 0.4 \quad (+5)$$

$$u_{5,1} = \frac{1}{3}u_{4,1} + \frac{2501}{37500} \quad (+1)$$

③
$$\begin{cases} u_{1,1} = \frac{1}{6}u_{0,1} + \frac{1}{6}u_{2,1} + \frac{2}{15} \\ u_{2,1} = \frac{1}{6}u_{1,1} + \frac{1}{6}u_{3,1} + \frac{7}{30} \\ u_{3,1} = \frac{1}{6}u_{2,1} + \frac{1}{6}u_{4,1} + \frac{7}{30} \\ u_{4,1} = \frac{1}{6}u_{3,1} + \frac{1}{6}u_{5,1} + \frac{2}{15} \end{cases} \quad (+4)$$

4. (15 points) The least squares approximation of a function $f(x)$ on an interval $[a, b]$ by a function

$$F_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots a_m y_m(x)$$

requires the determination of the coefficients $a_0, \cdots, a_m$ such that

$$\int_a^b [f(x) - F_m(x)]^2 dx$$

becomes minimum. Show that a necessary condition for that minimum is given by $m+1$ normal equations ($j = 0, \cdots, m$)

$$\sum_{k=0}^{m} h_{jk} a_k = b_j,$$

where

$$h_{jk} = \int_a^b y_j(x) y_k(x) dx, \; b_j = \int_a^b f(x) y_j(x) dx.$$

$$E(a_0, \cdots, a_m) = \int_a^b [f(x) - F_m(x)]^2 dx \quad +2$$

$$= \int_a^b f(x)^2 dx - 2\int_a^b f(x) F_m(x) dx + \int_a^b F_m(x)^2 dx \quad +2$$

$$= \int_a^b f(x)^2 dx - 2\sum_{j=0}^{m} a_j \int_a^b f(x) y_j(x) dx$$

$$+ \sum_{i=0}^{m}\sum_{j=0}^{m} a_i a_j \int_a^b y_i(x) y_j(x) dx \quad +3$$

$$= \int_a^b f(x)^2 dx - 2\sum_{j=0}^{m} a_j b_j + \sum_{i=0}^{m}\sum_{j=0}^{m} a_i a_j h_{ij}$$

$$\frac{\partial E}{\partial a_j} = -2b_j + \sum_{k=0}^{m} a_k h_{jk} + \sum_{i=0}^{m} a_i h_{ij} = 0 \quad +4$$

Since $h_{ij} = h_{ji}$,

$$\sum_{k=0}^{m} a_k h_{jk} + \sum_{k=0}^{m} a_k h_{jk} = 2b_j \quad +3$$

$$\therefore \sum_{k=0}^{m} h_{jk} a_k = b_j \text{ for } j = 0, \cdots, m$$

$+1$

5. (20 points) On each interval $x_j \le x \le x_{j+1}$, we define a cubic polynomial $p_j(x)$ such that

$$p_j(x_j) = f(x_j),\ p_j(x_{j+1}) = f(x_{j+1}),\ p_j'(x_j) = k_j,\ p_j'(x_{j+1}) = k_{j+1}.$$

(a) (7 points) Show that the following polynomial satisfies the above conditions:

$$\begin{aligned} p_j(x) &= f(x_j)c_j^2(x-x_{j+1})^2[1+2c_j(x-x_j)] + f(x_{j+1})c_j^2(x-x_j)^2[1-2c_j(x-x_{j+1})] \\ &\quad + k_j c_j^2(x-x_j)(x-x_{j+1})^2 + k_{j+1}c_j^2(x-x_j)^2(x-x_{j+1}), \end{aligned}$$

where $c_j = \frac{1}{x_{j+1}-x_j}$.

(b) (10 points) Show that

$$\begin{aligned} p_j''(x_j) &= -6c_j^2 f(x_j) + 6c_j^2 f(x_{j+1}) - 4c_j k_j - 2c_j k_{j+1} \\ p_j''(x_{j+1}) &= 6c_j^2 f(x_j) - 6c_j^2 f(x_{j+1}) + 2c_j k_j + 4c_j k_{j+1}. \end{aligned}$$

(c) (3 points) Show that the condition $p_{j-1}''(x_j) = p_j''(x_j)$ implies

$$c_{j-1}k_{j-1} + 2(c_{j-1}+c_j)k_j + c_j k_{j+1} = 3[c_{j-1}^2 \nabla f_j + c_j^2 \nabla f_{j+1}],$$

where $\nabla f_j = f(x_j) - f(x_{j-1})$ and $\nabla f_{j+1} = f(x_{j+1}) - f(x_j)$.

(a) $P_j(x_j) = f(x_j)\, c_j^2 (x_j - x_{j+1})^2 = f(x_j)$

$P_j(x_{j+1}) = f(x_{j+1})\, c_j^2 (x_{j+1} - x_j)^2 = f(x_{j+1})$ +1

$$\begin{aligned} P_j'(x) &= 2f(x_j)\, c_j^2 (x-x_{j+1})[1+2c_j(x-x_j)] + 2f(x_j)\, c_j^3 (x-x_{j+1})^2 \\ &\quad + 2f(x_{j+1})\, c_j^2 (x-x_j)[1-2c_j(x-x_{j+1})] - 2f(x_{j+1})\, c_j^3 (x-x_j)^2 \\ &\quad + k_j c_j^2 (x-x_{j+1})^2 + 2k_j c_j^2 (x-x_j)(x-x_{j+1}) \\ &\quad + k_{j+1} c_j^2 (x-x_j)^2 + 2k_{j+1} c_j^2 (x-x_j)(x-x_{j+1}) \end{aligned}$$ +4

$P_j'(x_j) = 2f(x_j)\cdot(-c_j) + 2f(x_j)\cdot c_j + k_j = k_j$ +1

$P_j'(x_{j+1}) = 2f(x_{j+1})\cdot c_j - 2f(x_{j+1})\cdot c_j + k_{j+1} = k_{j+1}$ +1

(c) $-6c_j^2 f(x_j) + 6c_j^2 f(x_{j+1}) - 4c_j k_j - 2c_j k_{j+1}$

$= 6c_{j-1}^2 f(x_{j-1}) - 6c_{j-1}^2 f(x_j) + 2c_{j-1}k_{j-1} + 4c_{j-1}k_j$ +1

$2c_{j-1}k_{j-1} + 4(c_{j-1}+c_j)k_j + 2c_j k_{j+1} = 6c_{j-1}^2[f(x_j) - f(x_{j-1})] + 6c_j^2[f(x_{j+1}) - f(x_j)]$

$c_{j-1}k_{j-1} + 2(c_{j-1}+c_j)k_j + c_j k_{j+1} = 3[c_{j-1}^2 \nabla f_j + c_j^2 \nabla f_{j+1}]$ +2

(Continued)

(b)
$$\begin{aligned} P_j''(x) = {} & 2f(x_j)\,c_j^2\left[1+2c_j(x-x_j)\right] + 8f(x_j)\,c_j^3(x-x_{j+1}) \\ & + 2f(x_{j+1})\,c_j^2\left[1-2c_j(x-x_{j+1})\right] - 8f(x_{j+1})\,c_j^3(x-x_j) \\ & + 4k_j\,c_j^2(x-x_{j+1}) + 2k_j\,c_j^2(x-x_j) \\ & + 4k_{j+1}\,c_j^2(x-x_j) + 2k_{j+1}\,c_j^2(x-x_{j+1}) \end{aligned}$$

+4

$$\begin{aligned} P_j''(x_j) = {} & 2f(x_j)\,c_j^2 - 8f(x_j)\,c_j^2 + 2f(x_{j+1})\,c_j^2 + 4f(x_{j+1})\,c_j^2 \\ & - 4k_j\,c_j - 2k_{j+1}\,c_j \end{aligned}$$

+3

$$= -6c_j^2 f(x_j) + 6c_j^2 f(x_{j+1}) - 4c_jk_j - 2c_jk_{j+1}$$

$$\begin{aligned} P_j''(x_{j+1}) = {} & 2f(x_j)\cdot c_j^2 + 4f(x_j)\,c_j^2 + 2f(x_{j+1})\,c_j^2 - 8f(x_{j+1})\,c_j^2 \\ & + 2k_j\,c_j + 4k_{j+1}\,c_j \end{aligned}$$

+3

$$= 6c_j^2 f(x_j) - 6c_j^2 f(x_{j+1}) + 2c_jk_j + 4c_jk_{j+1}$$