Quiz #5 (CSE 400.001)
November 26, 2012 (Monday)

Name: ______________________ **ID No:** ______________________

1. (15 points) Solve the following problem from $x = 1$ to $x = 5$ with $h = 2.0$ using the Euler method:

$$x^2y'' - 2xy' - 3y = 0, \quad y(1) = 3.0, \ y'(1) = 6.0,$$

$$y'' - \frac{2}{x}y' - \frac{3}{x^2}y = 0$$

$$y_1 = y, \quad y_2 = y', \quad y_1(1) = 3.0, \ y_2(1) = 6.0$$

+3

$$\begin{cases} y_1' = y_2 \\ y_2' = \frac{2}{x}y_2 + \frac{3}{x^2}y_1, \end{cases} \quad \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 3.0 \\ 6.0 \end{bmatrix}$$

+3

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

+3

$$\begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h \cdot \begin{bmatrix} y_{2,n} \\ \frac{2}{x_n}y_{2,n} + \frac{3}{x_n^2}y_{1,n} \end{bmatrix}$$

$$\begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} + 2.0 \begin{bmatrix} y_{2,0} \\ \frac{2}{1} * y_{2,0} + \frac{3}{1^2} * y_{1,0} \end{bmatrix}$$

+3

$$= \begin{bmatrix} 3.0 \\ 6.0 \end{bmatrix} + 2.0 \begin{bmatrix} 6.0 \\ 2 * 6.0 + 3 * 3.0 \end{bmatrix} = \begin{bmatrix} 15.0 \\ 48.0 \end{bmatrix}$$

$$\begin{bmatrix} y_{1,2} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} + 2.0 \begin{bmatrix} y_{2,1} \\ \frac{2}{3} * y_{2,1} + \frac{3}{3^2} * y_{1,1} \end{bmatrix}$$

+3

$$= \begin{bmatrix} 15.0 \\ 48.0 \end{bmatrix} + 2.0 \begin{bmatrix} 48.0 \\ \frac{2}{3} * 48.0 + \frac{1}{3} * 15.0 \end{bmatrix} = \begin{bmatrix} 111.0 \\ 122.0 \end{bmatrix}$$

2. (15 points) Consider a laterally insulated metal bar of length 1 and satisfying the heat equation $u_t = u_{xx}$. Suppose that the ends of the bar kept at temperature $u(0,t) = u(1,t) = 0$ and the initial temperature in the bar is $f(x) = 4x^2$, if $0 \le x \le 0.5$, and $f(x) = 4(x-1)^2$, if $0.5 \le x \le 1$. Applying the Crank-Nicolson method with $h = 0.2$ and $k = 0.08$, find the temperature $u(x,t)$ in the bar for $0 \le t \le 0.16$.

$$\frac{1}{k}(u_{i,j+1} - u_{ij}) = \frac{1}{2h^2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j}) + \frac{1}{2h^2}(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1})$$ +2

Let $r = \frac{k}{h^2} = \frac{0.08}{0.04} = 2$, then we have

$$u_{i,j+1} - u_{ij} = (u_{i+1,j} - 2u_{ij} + u_{i-1,j}) + (u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1})$$ +3

$$\underline{3u_{i,j+1} - u_{i+1,j+1} - u_{i-1,j+1} = -u_{ij} + u_{i+1,j} + u_{i-1,j}}$$

$u_{10} = 0.16$, $u_{20} = 0.64$, $u_{30} = u_{20}$, $u_{40} = u_{10}$ +2

For $j=0$:

$(i=1)$: $3u_{11} - u_{21} - \cancel{u_{01}} \overset{0}{=} -u_{10} + u_{20} + \cancel{u_{00}}^{0} = 0.48$ +2

$(i=2)$: $-u_{11} + 3u_{21} - \underset{u_{31}}{\underset{\parallel}{u_{21}}} = -u_{20} + \underset{u_{20}}{\underset{\parallel}{u_{30}}} + u_{10} = 0.16$

$\Rightarrow \underline{u_{11} = 0.224,\ u_{21} = 0.192}$ +2

For $j=1$:

$(i=1)$: $3u_{12} - u_{22} - \cancel{u_{02}}^{0} = -u_{11} + u_{21} + \cancel{u_{01}}^{0} = -0.032$

$(i=2)$: $-u_{12} + 2u_{22} = -u_{21} + \underset{u_{21}}{\underset{\parallel}{u_{31}}} + u_{11} = 0.224$

+2

$\Rightarrow \underline{u_{12} = 0.032,\ u_{22} = 0.128}$ +2