Chap 1. The Bare Basics

1.1 Points and Vectors

Coordinate system: origin $\mathbb{O}$ (a point) and
directions $\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3$ (vectors)

Ex:

$$\mathbb{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbb{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbb{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbb{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Point: a 2D/3D location
- Affine space (or Euclidean space):
  the collection of all 2D/3D points ($\mathbb{E}^2$ or $\mathbb{E}^3$)

Ex:

$$\mathbb{q} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad ; \quad \mathbb{P} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$q_x = -1, \ q_y = 2 \ ; \quad p_x = 1, \ p_y = 2, \ p_z = 4$$

- vector: the difference of two points

Ex:
$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbb{v} = \mathbb{P} - \mathbb{q} \quad (\mathbb{v}: \text{vector}; \ \mathbb{P}, \mathbb{q}: \text{points})$$

- $\mathbb{R}^2$ or $\mathbb{R}^3$: linear space or real space
  where vectors "live".
- Affine and linear spaces are not the same.
  But, every affine space has an associated linear space
  (formed by the differences of all point pairs).
  It is common to plot points and vectors together.

1.2 Operations

Translation: $\hat{\mathbb{p}} = \mathbb{p} + \mathbb{v}$

points ↳ vector

o. Translations change point coordinates, but they have no effect on vector coordinates.

Ex:

$\mathbb{a}, \mathbb{b}$: points, $\mathbb{w} = \mathbb{a} - \mathbb{b}$: vector

$\Rightarrow \mathbb{a}+\mathbb{v}, \mathbb{b}+\mathbb{v}$: translated points

$(\mathbb{a}+\mathbb{v}) - (\mathbb{b}+\mathbb{v}) = \mathbb{a} - \mathbb{b} = \mathbb{w}$: the same difference vector

Linear Combinations

$\mathbb{v} = \alpha_1 \mathbb{v}_1 + \alpha_2 \mathbb{v}_2 + \cdots + \alpha_n \mathbb{v}_n$, ($\alpha_i$: real number ; $\mathbb{v}_i$: vector)

o. Vectors may be combined using any real numbers as factors in a linear combination.
But, this is not true of points.

o. If we combine two points $\mathbb{p}$ and $\mathbb{q}$, to yield $\mathbb{x} = \alpha \mathbb{p} + \beta \mathbb{q}$, it is mandatory that $\alpha + \beta = 1$. → barycentric combination
Otherwise, the relation is not preserved under translation.

Ex 1.1

$$\mathbb{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbb{x} = 2\mathbb{p} + \mathbb{q} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now, translate by $\mathbb{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$$\hat{\mathbb{p}} = \mathbb{p} + \mathbb{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \hat{\mathbb{q}} = \mathbb{q} + \mathbb{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

But,

$$\hat{\mathbb{x}} = 2\hat{\mathbb{p}} + \hat{\mathbb{q}} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \neq \mathbb{x} + \mathbb{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Ex 1.2

Let $\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x} = 0.5\mathbf{p} + 0.5\mathbf{q} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

Translate by $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$$\hat{\mathbf{p}} = \mathbf{p} + \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \hat{\mathbf{q}} = \mathbf{q} + \mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

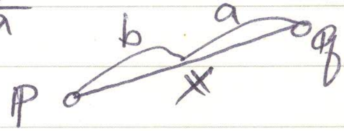
$$\hat{\mathbf{x}} = 0.5\hat{\mathbf{p}} + 0.5\hat{\mathbf{q}} = \begin{bmatrix} 2.5 \\ 3 \end{bmatrix} = \mathbf{x} + \mathbf{v} = \begin{bmatrix} 2.5 \\ 3 \end{bmatrix}!$$

o. Barycentric combination:

$$\mathbf{p} = \alpha_1\mathbf{p}_1 + \cdots + \alpha_n\mathbf{p}_n, \quad (\underbrace{\alpha_1 + \cdots + \alpha_n = 1}_{\text{partition of unity}})$$

o. Ratio of three points: (when $\mathbf{x} = a\mathbf{p} + b\mathbf{q}$) $\hookrightarrow a + b = 1$

$$\text{ratio}(\mathbf{p}, \mathbf{x}, \mathbf{q}) = b : a = \frac{b}{a}$$



1.3 Products

o. Dot product of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z \quad (\mathbf{v}, \mathbf{w} : \text{vectors})$$

$\nearrow$ scalar

o. Angle between $\mathbf{v}$ and $\mathbf{w}$:

$$\alpha = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}\right)$$

o. Length of $\mathbf{v}$: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

o. $\mathbf{v} \perp \mathbf{w}$ ($\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other)

$$\Leftrightarrow \mathbf{v} \cdot \mathbf{w} = 0 \Leftrightarrow \cos(\alpha) = 0 \Leftrightarrow \alpha = 90^\circ$$

o. Dot products are symmetric

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

- Cross product (or vector product) for 3D vectors:

$$\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}$$

- $(\mathbf{v} \wedge \mathbf{w}) \perp \mathbf{v}$ and $(\mathbf{v} \wedge \mathbf{w}) \perp \mathbf{w}$
- $\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin(\alpha)$:
  This length is equal to the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.
- $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$: cross products are antisymmetric.

1.4 Affine Maps

$\hat{\mathbf{p}} = A\mathbf{p} + \mathbf{v}$, where A : 3×3 matrix and $\mathbf{v} \in \mathbb{R}^3$

- $\mathbf{p} \in \mathbb{E}^3 \Rightarrow \hat{\mathbf{p}} \in \mathbb{E}^3$
- Affine maps map points to points, lines to lines, and planes to planes.
- Affine maps leave the ratio of three collinear points unchanged. $\Rightarrow$ a key characterization of affine map.

Ex 1.3

$\mathbf{p}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$: three collinear points

$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: affine map

$\Rightarrow \hat{\mathbf{p}}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\hat{\mathbf{p}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{\mathbf{p}}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$: collinear points

- $\mathbf{p}_2$ is the midpoint of $\mathbf{p}_1$ and $\mathbf{p}_3$
  $\Rightarrow \hat{\mathbf{p}}_2$ is the midpoint of $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_3$

Properties

- Affine maps take parallel lines to parallel lines
- If the lines don't intersect, they will not intersect after affine mapping.
- The same applies to planes.

1.5 Triangles and Tetrahedra

o. Area of a 2D triangle T formed by $\mathbb{a}, \mathbb{b}, \mathbb{c}$ ← three noncollinear points

$$\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{c}) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \mathbb{a} & \mathbb{b} & \mathbb{c} \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{vmatrix}$$

o. $\mathbb{p} \in T \Rightarrow \mathbb{p} = \underbrace{u\mathbb{a} + v\mathbb{b} + w\mathbb{c}}_{\text{a combination of points}}, \ (u+v+w=1)$

Using Cramer's rule,

$$u = \frac{\text{area}(\mathbb{p}, \mathbb{b}, \mathbb{c})}{\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{c})}, \quad v = \frac{\text{area}(\mathbb{a}, \mathbb{p}, \mathbb{c})}{\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{c})}, \quad w = \frac{\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{p})}{\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{c})}$$

o. $\mathbb{u} = (u, v, w)$: barycentric coordinates.

u, v, w are not independent each other ($w = 1-u-v$),

Nevertheless, they behave much like "normal" coordinates

o. If $\mathbb{p}$ is given, then we can find $\mathbb{u}$.

If $\mathbb{u}$ is given, then we can also find $\mathbb{p}$.

o. It is not necessary that $\mathbb{p}$ is inside T.

If it is not, then some of u, v, w will be negative.

⇒ The area calculation must produce a signed area.

o. Barycentric coordinates of the three vertices of T :

$\mathbb{a} \hat{=} (1,0,0)$, $\mathbb{b} \hat{=} (0,1,0)$, $\mathbb{c} \hat{=} (0,0,1)$

↳ symbol for the barycentric coordinates of a point

o. Area of a 3D triangle T formed by 3D points

$$\text{area}(\mathbb{a}, \mathbb{b}, \mathbb{c}) = \frac{1}{2}\| [\mathbb{b} - \mathbb{a}] \wedge [\mathbb{c} - \mathbb{a}] \|$$

o. Volume of a tetrahedron formed by four 3D points $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4$

$$\text{vol}(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4) = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \mathbb{P}_1 & \mathbb{P}_2 & \mathbb{P}_3 & \mathbb{P}_4 \end{vmatrix}.$$

Ex 1.4

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \text{area}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \frac{1}{2} \| [\mathbf{b} - \mathbf{a}] \wedge [\mathbf{c} - \mathbf{a}] \| \\ &= \frac{1}{2} \left\| \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\| \\ &= \frac{1}{2} \left\| \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix} \right\| = \frac{\sqrt{69}}{2} \end{aligned}$$

Ex 1.5

$$\mathbf{P}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{P}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{vol}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = \frac{12}{6} = 2$$