Chap 13. NURBS

13.1 Conics

A number of equivalent ways exist to define a conic section; for our purposes the following one is very useful: *A conic section in $\mathbb{E}^2$ is the perspective projection of a parabola in $\mathbb{E}^3$.*

When it comes to the formulation of conics as rational curves, one typically chooses the center of the projection to be the origin $\mathbf{0}$ of a 3D Cartesian coordinate system. The plane into which one projects is taken to be the plane $z = 1$. Since we will study planar curves in this section, we may think of this plane as a copy of $\mathbb{E}^2$. Our special projection is characterized by

$$\underline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} x/z \\ y/z \end{bmatrix} = \mathbf{x}.$$

Note that a point $\mathbf{x}$ is the projection of a whole family of points: Every point of the form $f\underline{\mathbf{x}}$ projects onto the same 2D point $\mathbf{x}$. The 3D point $\underline{\mathbf{x}}$ is called the *homogeneous form* or *homogeneous coordinates* of $\mathbf{x}$. Sketch 102 illustrates.

A conic $\mathbf{c}(t)$ is given by *weights* $z_0, z_1, z_2 \in \mathbb{R}$ and *control points* $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{E}^2$ such that

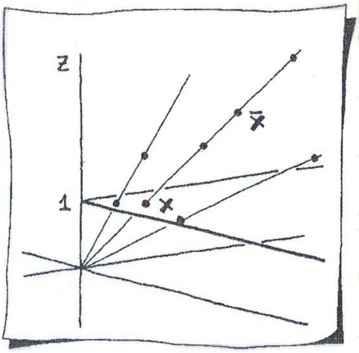
$$\mathbf{c}(t) = \frac{z_0\mathbf{b}_0 B_0^2(t) + z_1\mathbf{b}_1 B_1^2(t) + z_2\mathbf{b}_2 B_2^2(t)}{z_0 B_0^2(t) + z_1 B_1^2(t) + z_2 B_2^2(t)}, \tag{13.1}$$

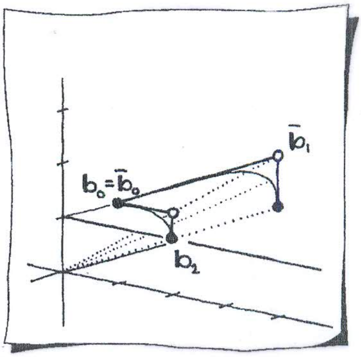
i.e., $\mathbf{c}$ may be expressed as a *parametric rational quadratic curve.*

Thus the conic control polygon is the projection of the control polygon with homogenoues vertices

$$z_0\begin{bmatrix} \mathbf{b}_0 \\ 1 \end{bmatrix}, \quad z_1\begin{bmatrix} \mathbf{b}_1 \\ 1 \end{bmatrix}, \quad z_2\begin{bmatrix} \mathbf{b}_2 \\ 1 \end{bmatrix},$$

which is the control polygon of the 3D parabola that we projected onto the conic $\mathbf{c}$. The form (13.1) is called the *rational quadratic form* of a conic section.





13.2 Reparametrization and Classification

It is possible to change the weights of a conic without changing its shape. If the initial weights are z_0, z_1, z_2, then the set of weights $z_0, cz_1, c^2 z_2$ generates the same conic for any $c \neq 0$. This may be used to bring a conic into *standard form*: Assuming $z_0 = 1$ without loss of generality, we set $c = 1/\sqrt{z_2}.^1$, Now the new weights are $1, cz_1, 1$.

Changing the weights in this fashion does not change the curve's geometry, but it does change how it is traversed. Hence, the term *reparametrization* is used to describe this process.

Once a conic is in standard form, it is easy to decide which type it is:

- a hyperbola if $z_1 > 1$;
- a parabola if $z_1 = 1$;
- an ellipse if $z_1 < 1$.

Figure 13.2 shows some examples. The "flat" segments are ellipses, while the "curved" ones are hyperbolas. The intermediate case, the parabola, is plotted in gray.

An interesting effect occurs if we set $c = -1$. Then, the weights z_0, z_1, z_2 change to new weights $z_0, -z_1, z_2$. While the first set of weights (assuming all z_i are positive) generates a curve inside the control polygon, the second set generates the remaining "half" of the curve, called the *complementary segment*. This comes in handy if we want to plot a whole conic: Simply plot a conic arc for each set of weights. This is done in Figure 13.3.

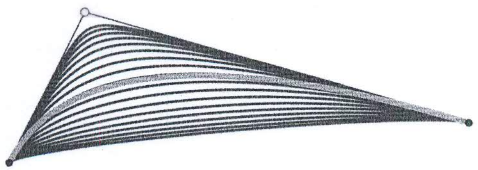


Figure 13.2.
As the weight z_1 changes form 0.1 to 0.9, three types of conics are produced.

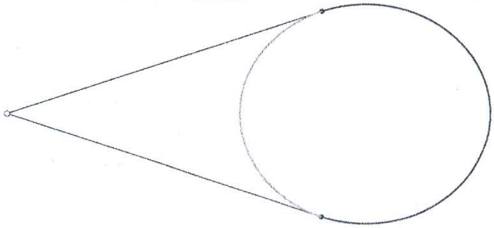


Figure 13.3.
An arc of a conic, grey, and the complementary segment, black.

13.4 The Circle

Of all conics, the circular arc is the one most widely used. Here we will represent it as a rational quadratic Bézier curve. Its control polygon must satisfy a special condition: It has to form an isosceles triangle, due to the circle's symmetry properties. Referring to Figure 13.4, and assuming standard form, we need to set

$$z_1 = \cos \alpha,$$

with $\alpha = \angle(\mathbf{b}_2, \mathbf{b}_0, \mathbf{b}_1)$.

A whole circle may be represented in many ways by piecewise rational quadratics. One example is to represent one quarter with the control polygon, and then use the complementary segment to write the remaining part. It is probably more convenient—retaining the convex hull property for positive weights—to dissect the full circle into four parts, as shown in Figure 13.5.

Although we can write an arc of a circle in rational quadratic form, one should not overlook that we then sacrifice one nice property of the familiar sin/cos parametrization: In the rational quadratic form, the parameter t does not traverse the circle with *unit speed.* Thus, if an arc of a rational quadratic is to be split into a certain number of segments, each subtending the same angle, numerical techniques must be invoked. In the sin/cos parametrization, by contrast, equal parameter increments ensure equal subtended angles.

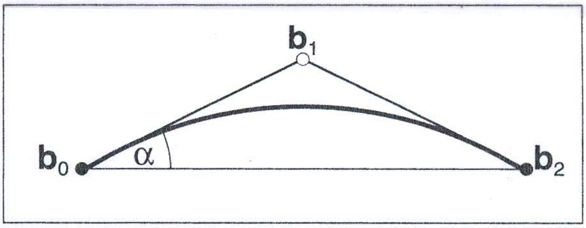


Figure 13.4.
The circle: the geometry of the control polygon.

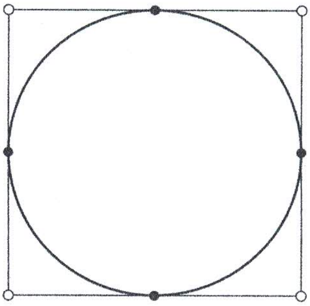


Figure 13.5.
The full circle: It may be represented by four rational quadratics.

13.5 Rational Bézier Curves

So far, we have obtained a conic section in $\mathbb{E}^2$ as the projection of a parabola (a quadratic) in $\mathbb{E}^3$. Conic sections may be expressed as rational quadratic Bézier curves, and their generalization to higher degree rational curves is quite straightforward: A rational Bézier curve of degree n in $\mathbb{E}^3$ is the projection of an n^{th} degree Bézier curve in $\mathbb{E}^4$ into the hyperplane $w = 1$. Here, we denote 4D points by four coordinates and their 3D projections by three coordinates:

$$\underline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix} = \mathbf{x}.$$

We may view this 4D hyperplane as a copy of $\mathbb{E}^3$; we assume that a point $\underline{\mathbf{x}}$ in $\mathbb{E}^4$ is given by four coordinates. Proceeding in exactly the same way as we did for conics, we can show that an n^{th} degree rational Bézier curve is given by

$$\mathbf{x}(t) = \frac{w_0\mathbf{b}_0B_0^n(t) + \cdots + w_n\mathbf{b}_nB_n^n(t)}{w_0B_0^n(t) + \cdots + w_nB_n^n(t)}; \quad \mathbf{x}(t), \mathbf{b}_i \in \mathbb{E}^3. \qquad (13.4)$$

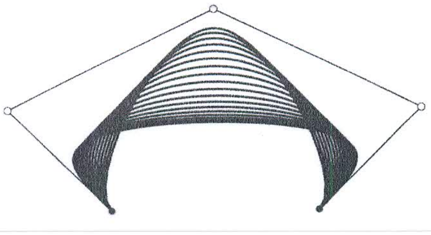
The w_i are again called *weights*; the $\mathbf{b}_i$ form the control polygon. It is the projection of the 4D control polygon $\underline{\mathbf{b}}_0, \ldots, \underline{\mathbf{b}}_n$. This 4D control polygon defines a 4D polynomial curve—the homogeneous form of the curve. It is given by

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{b}}_0B_0^n(t) + \cdots + \underline{\mathbf{b}}_nB_n^n(t).$$

In order to evaluate a rational Bézier curve, we apply the de Casteljau algorithm to this homogeneous form and project the resulting point into 3D.

If all weights are one, we obtain the standard nonrational Bézier curve; in that case, the denominator is identically equal to one. If some w_i are negative, singularities may occur; we will therefore only deal with nonnegative w_i. Rational Bézier curves enjoy all the properties that their nonrational counterparts possess; for example, they are affinely invariant. If all w_i are nonnegative, we have the convex hull property.[2]

The influence of the weights is illustrated in Figure 13.6. The "top" curve corresponds to $w_2 = 10$; the "bottom" one corresponds to $w_2 = 0.1$.



EXAMPLE 13.3

We will evaluate the following Bézier curve $t = 0.5$. Take the control points from Example 3.3; They are

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

However, make their weights $1, 2, 1, 1$. This gives the homogeneous control points

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Applying the de Casteljau algorithm to the homogeneous control points gives

$$\underline{\mathbf{x}}(0.5) = \begin{bmatrix} 0.0 \\ 0.375 \\ 1.375 \end{bmatrix}.$$

The corresponding 2D point is found after division by the third coordinate:

$$\mathbf{x}(0.5) = \begin{bmatrix} 0.0 \\ 0.2727 \end{bmatrix}.$$

Rational Bézier curves enjoy a property which is not shared by their nonrational brethren: This is *projective invariance.* A projective map maps homogeneous coordinates $\underline{\mathbf{x}}$ to new homogeneous coordinates $\bar{\underline{\mathbf{x}}}$. It takes the form of a linear map

$$\bar{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$$

with A being a 4×4 matrix. Such a map will change the weights of a curve. For the simple example of rational quadratic conics, projective maps are capable of mapping an ellipse to a hyperbola!

The curvature and torsion formulas from Section 8.2 change just slightly for rational curves. At $t = 0$ we have

$$\kappa(0) = 2\frac{n-1}{n}\frac{w_0 w_2}{w_1}\frac{\text{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2]}{\|\mathbf{b}_1 - \mathbf{b}_0\|^3}$$

and

$$\tau(0) = \frac{3}{2}\frac{n-2}{n}\frac{w_0 w_3}{w_1 w_2}\frac{\text{volume}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]}{\text{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2]^2}.$$

13.6 Rational B-Spline Curves

Rational B-spline curves, known as NURBS, short for *NonUniform Rational B-spline curveS* have become a standard in the CAD/CAM industry.[3] They are defined in a not too surprising fashion:

$$\mathbf{x}(u) = \frac{w_0\mathbf{d}_0 N_0^n(u) + \ldots + w_{D-1}\mathbf{d}_{D-1}N_{D-1}^n(u)}{w_0 N_0^n(u) + \ldots + w_{D-1}N_{D-1}^n(u)}. \qquad (13.5)$$

All properties from the rational Bézier form carry over, such as convex hull (for nonnegative weights), or affine and projective invariance.

Derivatives may easily be computed using the equations of Section 13.3.

Designing with cubic NURB curves is not very different from designing with their nonrational counterparts. But we now have the added freedom of being able to change weights. A change of only one weight affects a rational B-spline curve only locally.

13.7 Rational Bézier and B-Spline Surfaces

We can generalize Bézier and B-spline surfaces to their rational counterparts in much the same way as we did for the curve cases. In other words, we define a rational Bézier or B-spline surface as the projection of a 4D tensor product Bézier or B-spline surface. Thus, the rational Bézier patch takes the form

$$\mathbf{x}(u, v) = \frac{M^{\mathrm{T}}\mathbf{B}_w N}{M^{\mathrm{T}}WN}. \qquad (13.6)$$

The notation is that of (6.8), but now the matrix $\mathbf{B}_w$ has elements $w_{i,j}\mathbf{b}_{i,j}$ and the matrix W has elements $w_{i,j}$. These $w_{i,j}$ are again called weights and influence the shape of the surface in much the same way as we observed for the curve case.

A rational B-spline surface is similarly written as

$$\mathbf{s}(u, v) = \frac{M^{\mathrm{T}}\mathbf{D}_w N}{M^{\mathrm{T}}WN}, \qquad (13.7)$$

where the matrices M and N contain the B-spline basis functions in u and v.

13.8 Surfaces of Revolution

One advantage of rational B-spline surfaces is that they allow the exact representation of surfaces of revolution. A surface of revolution is obtained by rotating or sweeping a curve—the *generatrix*—around an axis. Our generatrix will be of the form

$$\mathbf{g}(v) = \begin{bmatrix} r(v) \\ 0 \\ z(v) \end{bmatrix}.$$

This planar curve in the (x, z)-plane would be a (rational) Bézier curve or a B-spline curve in most practical cases. For our axis of revolution, we will take the z-axis. In Figure 13.8, the z-axis comes out of the center of the half-torus. The (x, z)-plane is nearly aligned with the view.

In this context, a surface of revolution is given by

$$\mathbf{x}(u, v) = \begin{bmatrix} r(v)\cos u \\ r(v)\sin u \\ z(v) \end{bmatrix}$$

For fixed v, an isoparametric line $v = const$ traces out a circle of radius $r(v)$, called a *meridian*. Since a circle may be exactly represented by rational quadratic arcs, we may find an exact rational representation of a surface of revolution given that $r(v)$ and $z(v)$ are in rational form.

Let

$$\mathbf{c}_i = \begin{bmatrix} x_i \\ 0 \\ z_i \end{bmatrix}$$

be the control points of the generatrix and let w_i be their weights.[4] Then, the surface of revolution is broken down into four symmetric pieces which are rational quadratic in the parameter u. Each piece corresponds to one quadrant of the (x, y)-plane.

Over the first quadrant, we have a surface with three columns of control points and associated weights. They are given by

$$\begin{bmatrix} x_i \\ 0 \\ z_i \end{bmatrix}, \quad \begin{bmatrix} x_i \\ x_i \\ z_i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ x_i \\ z_i \end{bmatrix}.$$

Their weights are $w_i, \frac{\sqrt{2}}{2}w_i, w_i$. The remaining three surface segments are now simply obtained by reflecting this one appropriately.