Chap 5. Putting Curves to Work

5.1 Cubic Interpolation

Given four point/parameter pairs $(\mathbb{P}_i, t_i)$,
find a cubic Bézier curve $\mathbb{X}(t)$ such that

$$\mathbb{X}(t_i) = \mathbb{P}_i, \quad i = 0, 1, 2, 3.$$

Let

$$\mathbb{X}(t) = B_0^3(t)\mathbb{b}_0 + B_1^3(t)\mathbb{b}_1 + B_2^3(t)\mathbb{b}_2 + B_3^3(t)\mathbb{b}_3,$$

$$\Rightarrow \begin{bmatrix} \mathbb{P}_0 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \end{bmatrix} = \begin{bmatrix} B_0^3(t_0) & B_1^3(t_0) & B_2^3(t_0) & B_3^3(t_0) \\ B_0^3(t_1) & B_1^3(t_1) & B_2^3(t_1) & B_3^3(t_1) \\ B_0^3(t_2) & B_1^3(t_2) & B_2^3(t_2) & B_3^3(t_2) \\ B_0^3(t_3) & B_1^3(t_3) & B_2^3(t_3) & B_3^3(t_3) \end{bmatrix} \begin{bmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \mathbb{b}_2 \\ \mathbb{b}_3 \end{bmatrix}$$

$$\mathbb{P} = M\mathbb{B} \Rightarrow \mathbb{B} = M^{-1}\mathbb{P}$$

5.2 Interpolation Beyond Cubics

$$\mathbb{P} = M\mathbb{B}, \qquad M = (m_{i,j})_{(n+1)\times(n+1)} = (B_j^n(t_i))_{(n+1)\times(n+1)}$$

o. While polynomial interpolation is guaranteed to work, a small change in data can lead to large changes in the interpolating curve. $\Rightarrow$ ill-conditioned

o. Interpolating curve in monomial form:

$$\mathbb{X}(t) = \mathbb{a}_0 + \mathbb{a}_1 t + \cdots + \mathbb{a}_n t^n,$$

$$\mathbb{P} = MA, \text{ where } M = (m_{i,j})_{(n+1)\times(n+1)} = (t_i^j)_{(n+1)\times(n+1)}$$

The curve is the same as the above Bézier curve.

o. Interpolating curve using Lagrange polynomials:

$$L_i^n(t) = \frac{(t-t_0)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_n)}{(t_i-t_0)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_n)},$$

$$\mathbb{X}(t) = L_0^n(t)\mathbb{P}_0 + \cdots + L_n^n(t)\mathbb{P}_n : \text{the cardinal form.}$$

5.3 Aitken's Algorithm

o. This is a recursive algorithm to compute points on the interpolating polynomical curve; it has some characteristics of the de Casteljau algorithm.

o. Assume $\mathbb{P}_0^2(t)$: quadratic curve through $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$

$\mathbb{P}_1^2(t)$: — " — $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$.

$$\Rightarrow \mathbb{P}_0^3(t) = \frac{(t_3-t)}{(t_3-t_0)} \mathbb{P}_0^2(t) + \frac{(t-t_0)}{(t_3-t_0)} \mathbb{P}_1^2(t) : \text{interpolating cubic}$$

$$\left(\mathbb{P}_0^3(t_0) = \mathbb{P}_0^2(t_0) = \mathbb{P}_0 ; \ \mathbb{P}_0^3(t_1) = \frac{(t_3-t_1)}{(t_3-t_0)} \mathbb{P}_1 + \frac{(t_1-t_0)}{(t_3-t_0)} \mathbb{P}_1 = \mathbb{P}_1, \cdots \right)$$

o. How did we find the quadratic interpolants $\mathbb{P}_0^2(t), \mathbb{P}_1^2(t)$?

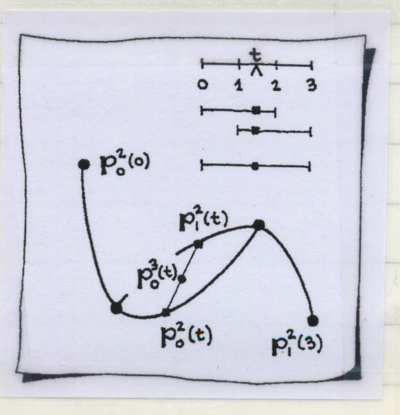
$$\mathbb{P}_0^2(t) = \frac{t_2-t}{t_2-t_0} \mathbb{P}_0^1(t) + \frac{t-t_0}{t_2-t_0} \mathbb{P}_1^1(t),$$

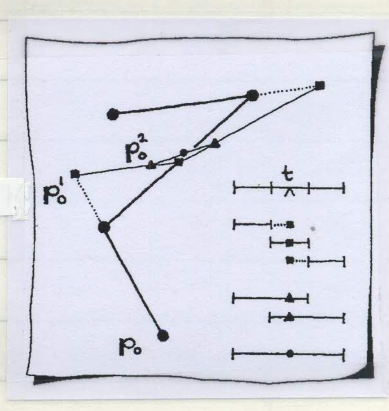
$$\mathbb{P}_1^2(t) = \frac{t_3-t}{t_3-t_1} \mathbb{P}_1^1(t) + \frac{t-t_1}{t_3-t_1} \mathbb{P}_2^1(t).$$

where $\mathbb{P}_1^1(t) = \frac{t_2-t}{t_2-t_1} \mathbb{P}_1 + \frac{t-t_1}{t_2-t_1} \mathbb{P}_2$, etc, ...

o. It is convenient to arrange the intermediate points as follows

$$\begin{array}{lllll}
\mathbb{P}_0 & & & & \\
 & \mathbb{P}_0^1(t) & & & \\
\mathbb{P}_1 & & \mathbb{P}_0^2(t) & & \\
 & \mathbb{P}_1^1(t) & & \mathbb{P}_0^3(t) & \\
\mathbb{P}_2 & & \mathbb{P}_1^2(t) & & \\
 & \mathbb{P}_2^1(t) & & & \\
\mathbb{P}_3 & & & &
\end{array}$$





5.4 Approximation

- When more data points should be interpolated, an approximation curve will be needed. (High degree interpolation becomes ill-conditioned.)
- Given $(\mathbb{P}_i, t_i)$, $i=0,\cdots,l$, we wish to find a polynomial curve $\mathbf{x}(t)$ of degree n so that the distances $\|\mathbb{P}_i - \mathbf{x}(t_i)\|$ are small. $(l>n)$

$$\mathbb{P} = M\mathbb{B}, \quad \text{where } M = \left(B_j^n(t_i)\right)_{(l+1)\times(n+1)}.$$

↳ overdetermined

$$\Rightarrow \quad M^T M \mathbb{B} = M^T \mathbb{P} \ : \text{normal equation}$$

$$\mathbb{B} = (M^T M)^{-1} M^T \mathbb{P} : \text{least squares solution}.$$

↳ invertible.

5.5 Finding the Right Parameters

- $t_i = i/l$: the uniform set of parameters.
- Chord length parameters:

$$\begin{cases} t_0 = 0 \\ t_1 = t_0 + \|\mathbb{P}_1 - \mathbb{P}_0\| \\ \quad \vdots \\ t_l = t_{l-1} + \|\mathbb{P}_l - \mathbb{P}_{l-1}\|. \end{cases}$$

- The parameters may be normalized : $t_i = \dfrac{t_i - t_0}{t_l - t_0}$.
- In general, the chord length parameters are superior to the uniform parameters.
- In the case of the ill-conditioned interpolation problem from Figure 5.2, the distorted data set with chord length parameters generates a curve that is indistinguishable from the true circle.

5.6 Hermite Interpolation

o. Given two points $\mathbb{P}_0, \mathbb{P}_1$ and two tangent vectors $\mathbf{v}_0, \mathbf{v}_1$, find a cubic polynomial curve $\mathbf{x}(t)$ so that

$$\mathbf{x}(0) = \mathbb{P}_0, \quad \mathbf{x}'(0) = \mathbf{v}_0, \quad \mathbf{x}'(1) = \mathbf{v}_1, \quad \mathbf{x}(1) = \mathbb{P}_1.$$

Let $\mathbf{x}(t) = B^3_0(t)\,\mathbf{b}_0 + B^3_1(t)\,\mathbf{b}_1 + B^3_2(t)\,\mathbf{b}_2 + B^3_3(t)\,\mathbf{b}_3$

$\Rightarrow$ $\mathbf{b}_0 = \mathbb{P}_0$ and $\mathbf{b}_3 = \mathbb{P}_1$

$\mathbf{b}_1 = \mathbb{P}_0 + \frac{1}{3}\mathbf{v}_0$ and $\mathbf{b}_2 = \mathbb{P}_1 - \frac{1}{3}\mathbf{v}_1$.

($\because$ $\mathbf{x}'(0) = 3\Delta\mathbf{b}_0$ and $\mathbf{x}'(1) = 3\Delta\mathbf{b}_2$)

$$\Rightarrow \mathbf{x}(t) = \mathbb{P}_0 B^3_0(t) + (\mathbb{P}_0 + \tfrac{1}{3}\mathbf{v}_0) B^3_1(t) + (\mathbb{P}_1 - \tfrac{1}{3}\mathbf{v}_1) B^3_2(t) + \mathbb{P}_1 B^3_3(t)$$

$$= \mathbb{P}_0 H^3_0(t) + \mathbf{v}_0 H^3_1(t) + \mathbf{v}_1 H^3_2(t) + \mathbb{P}_1 H^3_3(t)$$

where
$$\begin{cases} H^3_0(t) = B^3_0(t) + B^3_1(t) \\ H^3_1(t) = \frac{1}{3} B^3_1(t) \\ H^3_2(t) = -\frac{1}{3} B^3_2(t) \\ H^3_3(t) = B^3_2(t) + B^3_3(t) \end{cases}$$

Cubic Hermite polynomials

$\downarrow$

cardinal form

(the input data appear explicitly)

o. The lengths of $\mathbf{v}_0$ and $\mathbf{v}_1$ are important for the curve shape.

But, the lengths are not very intuitive to the user.