Quiz #1 (CSE4190.667)

March 28, 2012 (Wednesday)

Name: ______________ Dept: ______________ ID No: ______________

1. (10 points) Let four points be given by

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

Setting $t_i = i/3$, find the three control points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, for the quadratic Bézier curve $\mathbf{x}(t)$, $0 \le t \le 1$, that interpolates $\mathbf{x}(0) = \mathbf{p}_0$ and $\mathbf{x}(1) = \mathbf{p}_3$, and minimizes the following sum of squares:

$$\|\mathbf{x}(t_1) - \mathbf{p}_1\|^2 + \|\mathbf{x}(t_2) - \mathbf{p}_2\|^2.$$

$$\mathbf{b}_0 = \mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; (+1), \quad \mathbf{b}_2 = \mathbf{p}_3 = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \; (+1)$$

$$\mathbf{x}\left(\tfrac{1}{3}\right) = \tfrac{4}{9}\mathbf{b}_0 + \tfrac{4}{9}\mathbf{b}_1 + \tfrac{1}{9}\mathbf{b}_2 = \tfrac{4}{9}\mathbf{b}_1 + \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \; (+1)$$

$$\mathbf{x}\left(\tfrac{2}{3}\right) = \tfrac{1}{9}\mathbf{b}_0 + \tfrac{4}{9}\mathbf{b}_1 + \tfrac{4}{9}\mathbf{b}_2 = \tfrac{4}{9}\mathbf{b}_1 + \begin{bmatrix} 4/3 \\ 4 \end{bmatrix} \; (+1)$$

$$\frac{1}{9}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\mathbf{b}_1 = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \end{bmatrix} \; (+3)$$

$$\begin{bmatrix} 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix}\mathbf{b}_1 = \begin{bmatrix} 4 & 4 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 6 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 48 \\ 0 \end{bmatrix} \; (+2)$$

$$32\,\mathbf{b}_1 = \begin{bmatrix} 48 \\ 0 \end{bmatrix} \Rightarrow \mathbf{b}_1 = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \; (+1)$$

2. (10 points) What is the cubic Bézier form of the following 3D curve segment corresponding to $t \in [0, 1]$, when the curve is given in monomial form by

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ t \\ t^3 + 2t^2 + 3t + 4 \end{bmatrix}.$$

$$1 = [(1-t)+t]^3 = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 \quad +1$$

$$t = t[(1-t)+t]^2 = (1-t)^2 t + 2(1-t)t^2 + t^3 \quad +2$$

$$t^2 = t^2[(1-t)+t] = (1-t)t^2 + t^3 \quad +1$$

$$t^3 + 2t^2 + 3t + 4 = 4(1-t)^3 + 15(1-t)^2 t + 20(1-t)t^2 + 10t^3 \quad +2$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}}_{=\mathbf{b}_0 \; +1} (1-t)^3 + \underbrace{\begin{bmatrix} 1 \\ 1/3 \\ 5 \end{bmatrix}}_{=\mathbf{b}_1 \; +1} 3(1-t)^2 t + \underbrace{\begin{bmatrix} 1 \\ 2/3 \\ 20/3 \end{bmatrix}}_{=\mathbf{b}_2 \; +1} 3(1-t)t^2 + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 10 \end{bmatrix}}_{=\mathbf{b}_3 \; +1} t^3$$

3. (5 points) Degree elevate the following quadratic curve to a cubic Bézier curve by computing the four control points $\mathbf{b}_i$, for $i = 0, 1, 2, 3$:

$$\mathbf{x}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} (1-t)^2 + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} 2(1-t)t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2, \quad \text{for } 0 \le t \le 1.$$

$$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad +1$$

$$\mathbf{b}_1 = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \quad +2$$

$$\mathbf{b}_2 = \frac{2}{3}\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad +2$$

$$\therefore \mathbf{x}(t) = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{=\mathbf{b}_0} (1-t)^3 + \underbrace{\begin{bmatrix} 1/3 \\ 0 \end{bmatrix}}_{=\mathbf{b}_1} 3(1-t)^2 t + \underbrace{\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}}_{=\mathbf{b}_2} 3(1-t)t^2 + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{=\mathbf{b}_3} t^3$$