Offsets, Sweeps, and Minkowski sums for Freeform Curves and Surfaces

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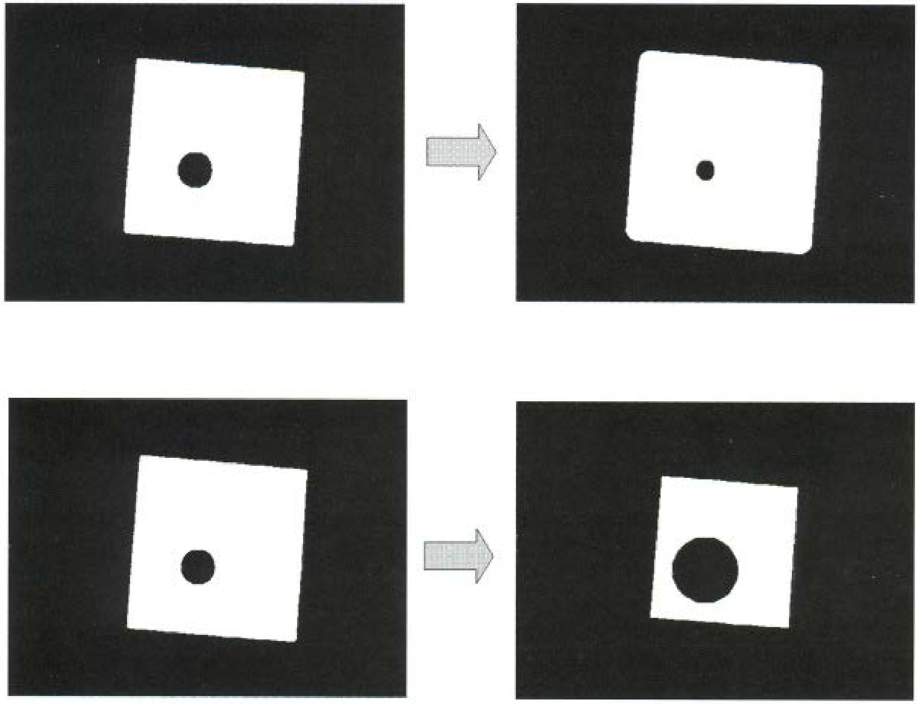
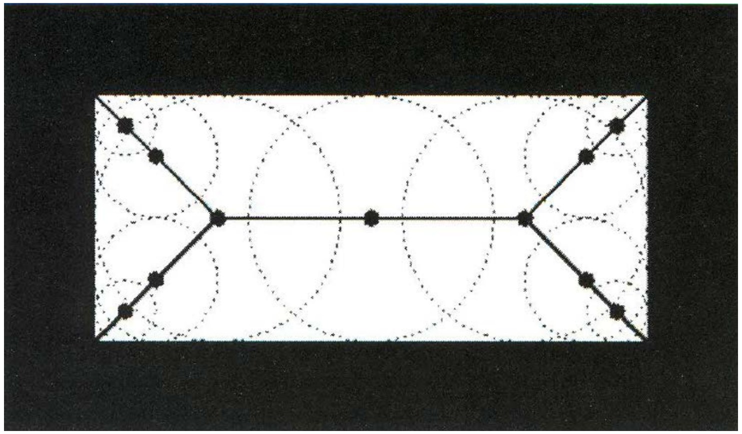
Motivation

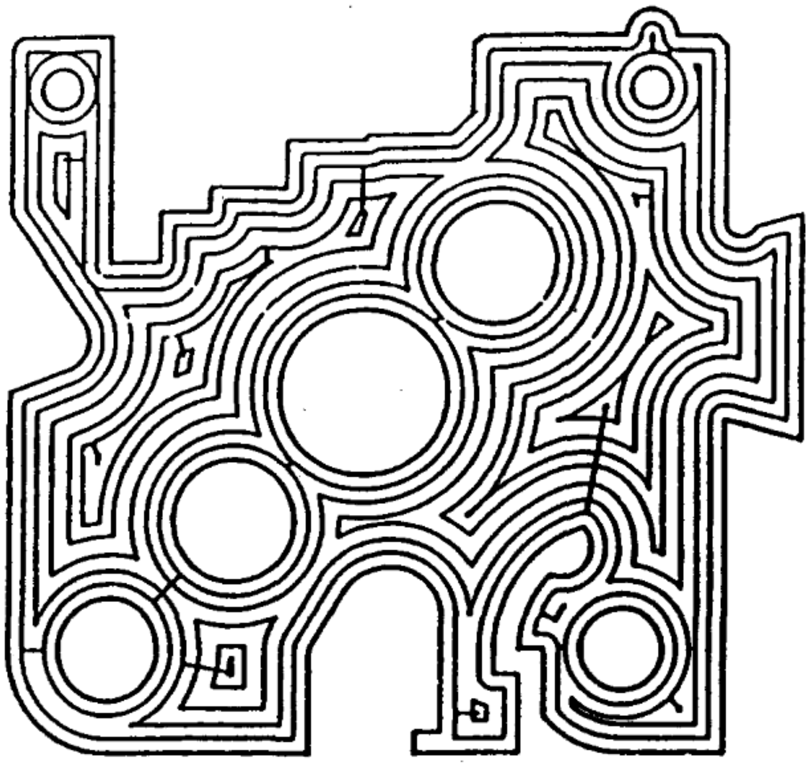
Dilation, Erosion, and Medial Axis

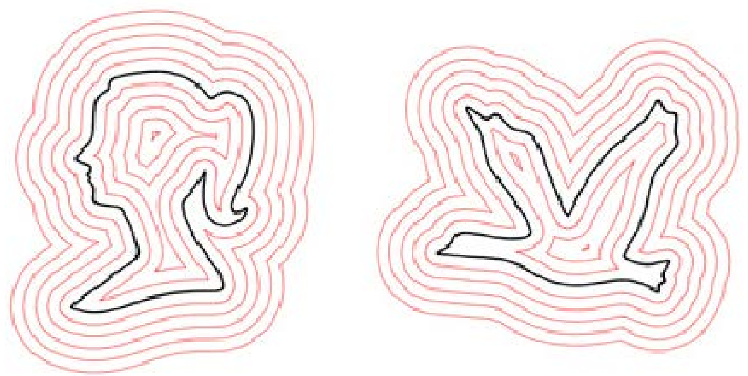




Motivation

Tool Path Generation for NC Machining



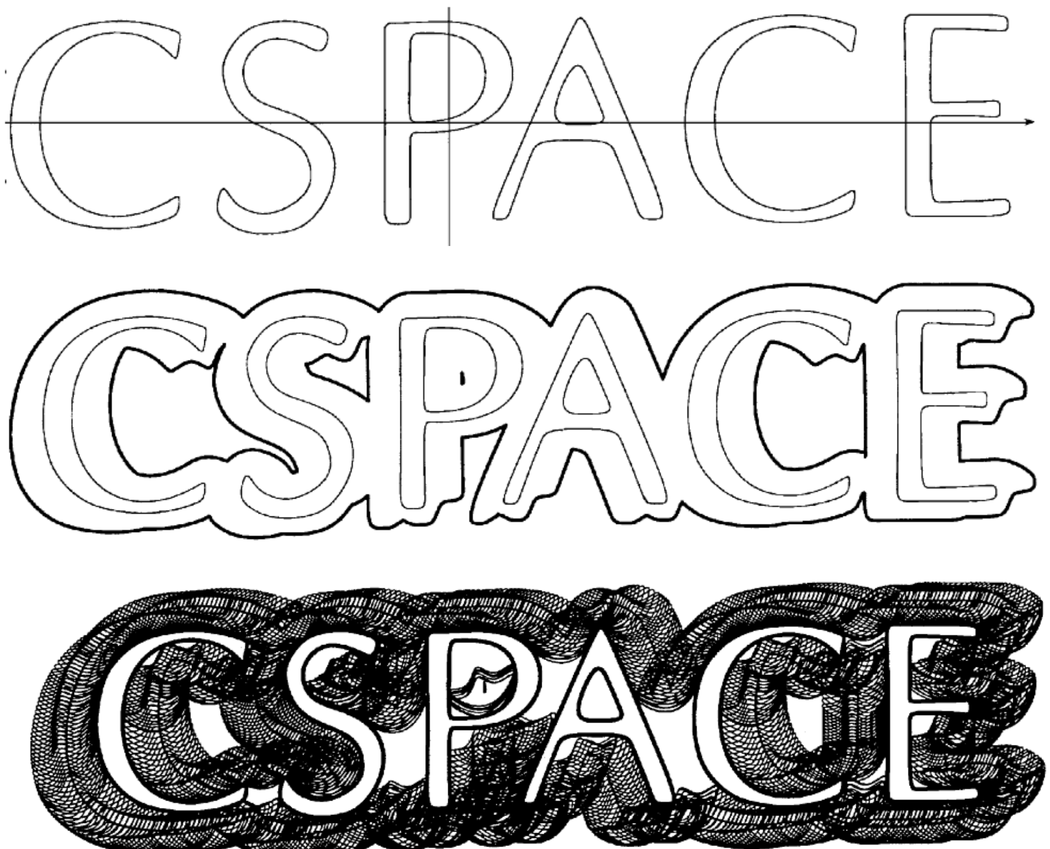


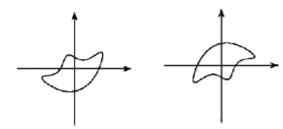
$$C_r(t) = C(t) + r \cdot N(t),$$

$$N(t) = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}.$$

Motivation

Collision Avoidance Motion Planning





$$A \cap (B + \mathbf{p}) \neq \emptyset$$

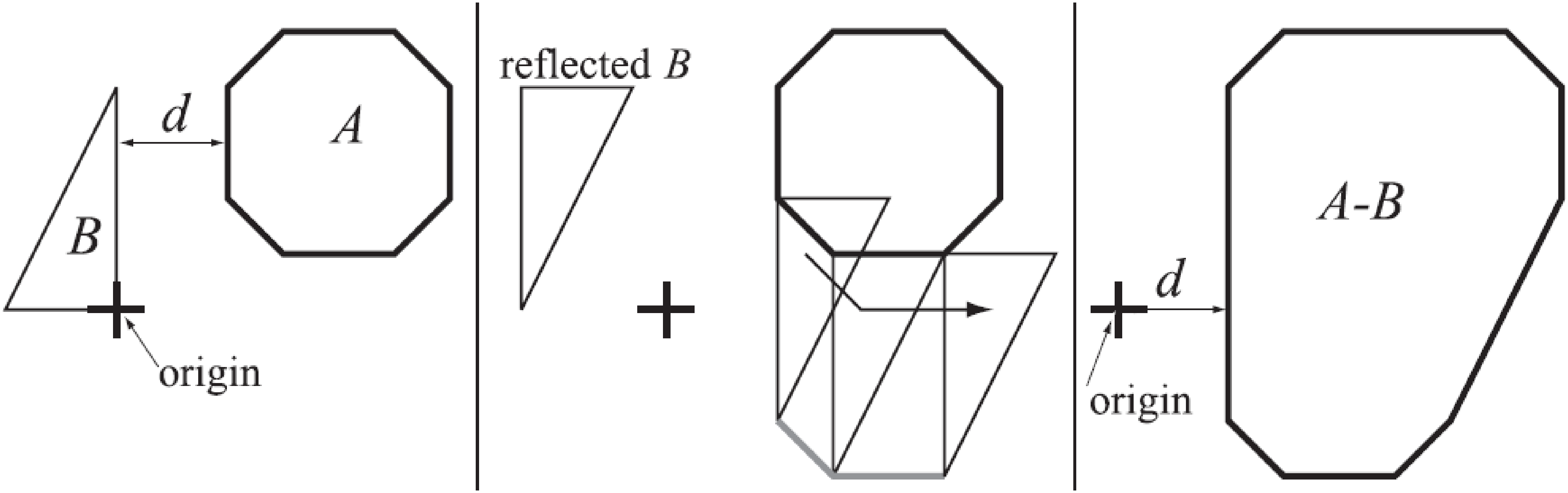
$$\mathbf{a} = \mathbf{b} + \mathbf{p}$$

$$\mathbf{p} = \mathbf{a} - \mathbf{b}$$

$$\mathbf{p} \in A - B$$

Motivation

Minimum Distance Computation

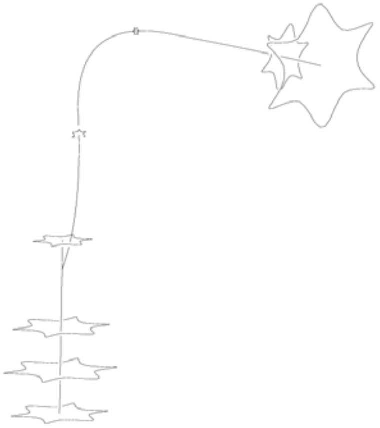
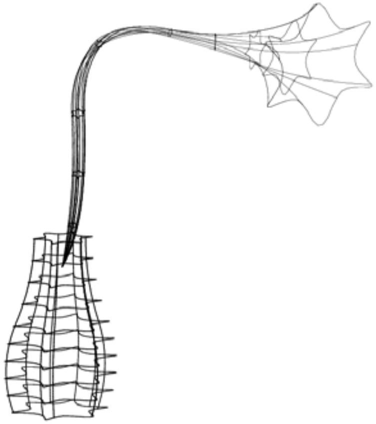


$$A - B = \{\mathbf{a} - \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$$

Motivation

Natural Shape Design and

Compact Shape Representation


In these special cases, the sweep surfaces are rational

Outline

Introduction

Research Issues

Non-rational Envelope of Curves and Surfaces

Rational Envelope of Lines and Planes

Offset Trimming

Conclusions

Introduction

- Conventional Research in CAGD

  Design and representation of freeform geometry

- Geometric Operations

  Offsets, Minkowski sums, sweeps

  Medial axis transformation, bisectors

  Voronoi diagrams and Voronoi cells

- Fundamental Difficulties

  Results are often non-rational curves/surfaces

  Arrangement of algebraic varieties

  High degree, robustness, efficiency, etc

Definitions

Offset of A

$$A \uparrow r = \cup_{\mathbf{p} \in A} O_r(\mathbf{p})$$

Minkowski Sum of A and B

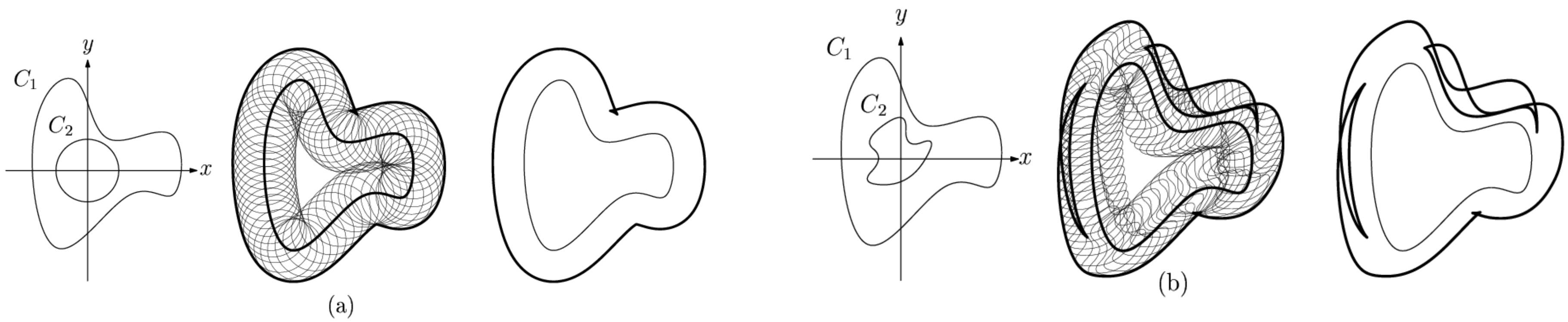
$$\begin{aligned} A \oplus B &= \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\} \\ &= \cup_{\mathbf{a} \in A} (B + \mathbf{a}) \\ &= \cup_{\mathbf{b} \in B} (A + \mathbf{b}) \end{aligned}$$

$$A \uparrow r = A \oplus O_r(\mathbf{0})$$

Envelope Curve

$C_1(u) = (x_1(u), y_1(u))$: Trajectory
$C_2(v) = (x_2(v), y_2(v))$: Moving curve
$(x(u,v), y(u,v))$: Envelope curve defined by

$$\begin{aligned} x(u,v) &= x_1(u) + x_2(v), \\ y(u,v) &= y_1(u) + y_2(v), \\ F(u,v) &= x_1'(u)y_2'(v) - y_1'(u)x_2'(v) = 0. \end{aligned}$$



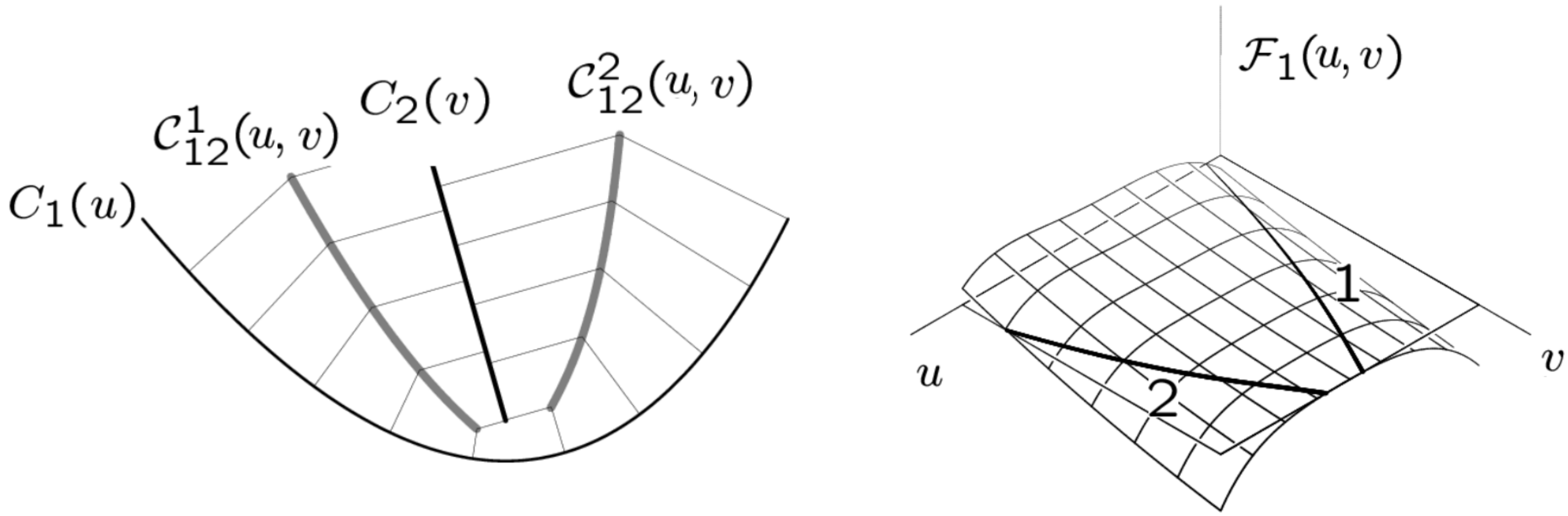
Envelope Curve Equation

$$\begin{aligned} x &= x_1(u) + x_2(v) \\ y &= y_1(u) + y_2(v) \\ 0 &= x_1'(u)y_2'(v) - y_1'(u)x_2'(v) \end{aligned}$$

Eliminating u and v, the envelope curve $e(x, y) = 0$ has algebraic degree $O(d^3)$ much higher than $(2d - 2)$ of $F(u, v) = 0$.

Bisector Curves

$$\left\langle (x,y) - C_1(u), C_1'(u) \right\rangle = 0,$$
$$\left\langle (x,y) - C_2(v), C_2'(v) \right\rangle = 0,$$
$$\left\langle (x,y) - \frac{C_1(u) + C_2(v)}{2}, C_1(u) - C_2(v) \right\rangle = 0.$$



Bisector Equation

$$\left\langle (x,y) - C_1(u), C_1'(u) \right\rangle = 0,$$
$$\left\langle (x,y) - C_2(v), C_2'(v) \right\rangle = 0,$$
$$\left\langle (x,y) - \frac{C_1(u) + C_2(v)}{2}, C_1(u) - C_2(v) \right\rangle = 0.$$

Eliminating u and v, the curve $b(x,y) = 0$ has degree $7d_1d_2 - 3(d_1 + d_2) + 1$

Eliminating x and y, we have $F(u,v) = 0$ of degree $2(d_1 + d_2) - 2$.

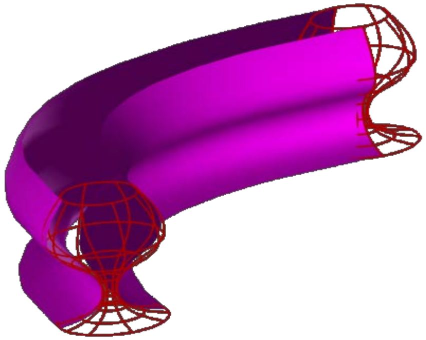
For $d_1 = d_2 = 3$, $F(u,v) = 0$ has degree 10 and $b(x,y) = 0$ has degree 46

Sweep Envelope Surface

$$S(u,v) = (s_1(u,v), s_2(u,v), s_3(u,v))^T$$

$$\begin{aligned} & T(u,v,t) \\ = \; & (x(u,v,t), y(u,v,t), z(u,v,t))^T \\ = \; & \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} \begin{bmatrix} s_1(u,v) \\ s_2(u,v) \\ s_3(u,v) \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix} \end{aligned}$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix} = 0$$



Calculus on Envelope Curve

$$x = x_1(u) + x_2(v),$$
$$y = y_1(u) + y_2(v),$$
$$F(u,v) = x_1'(u)y_2'(v) - y_1'(u)x_2'(v) = 0.$$

$$F(u,v) = 0$$
$$F_u + F_v \frac{dv}{du} = 0$$
$$F_{uu} + 2F_{uv}\frac{dv}{du} + F_{vv}\left(\frac{dv}{du}\right)^2 + F_v \frac{d^2v}{du^2} = 0$$

$$\frac{dv}{du} = -\frac{F_u}{F_v}$$
$$\frac{d^2v}{du^2} = -\frac{F_{uu} + 2F_{uv}\frac{dv}{du} + F_{vv}\left(\frac{dv}{du}\right)^2}{F_v}$$
$$= \frac{-F_{uu}F_v^2 + 2F_{uv}F_uF_v - F_{vv}F_u^2}{F_v^3}$$

Calculus on Envelope Curve

$$\frac{dx}{du} = x_u(u,v) + x_v(u,v)\frac{dv}{du}$$

$$\frac{dy}{du} = y_u(u,v) + y_v(u,v)\frac{dv}{du}$$

$$x = x_1(u) + x_2(v),$$
$$y = y_1(u) + y_2(v),$$
$$F(u,v) = x_1'(u)y_2'(v) - y_1'(u)x_2'(v) = 0.$$

$$\frac{d^2x}{du^2} = x_{uu} + 2x_{uv}\frac{dv}{du} + x_{vv}\left(\frac{dv}{du}\right)^2 + x_v\frac{d^2v}{du^2}$$

$$\frac{d^2y}{du^2} = y_{uu} + 2y_{uv}\frac{dv}{du} + y_{vv}\left(\frac{dv}{du}\right)^2 + y_v\frac{d^2v}{du^2}$$

$$\kappa(u) = \frac{\frac{dx}{du}\frac{d^2y}{du^2} - \frac{d^2x}{du^2}\frac{dy}{du}}{\left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2\right]^{3/2}}$$

Envelope Surface

Given two surfaces

$S_1(u,v) = (x_1(u,v), y_1(u,v), z_1(u,v))$

$S_2(s,t) = (x_2(s,t), y_2(s,t), z_2(s,t))$

Envelope surface $(x(u,v,s,t), y(u,v,s,t), z(u,v,s,t))$ is defined by

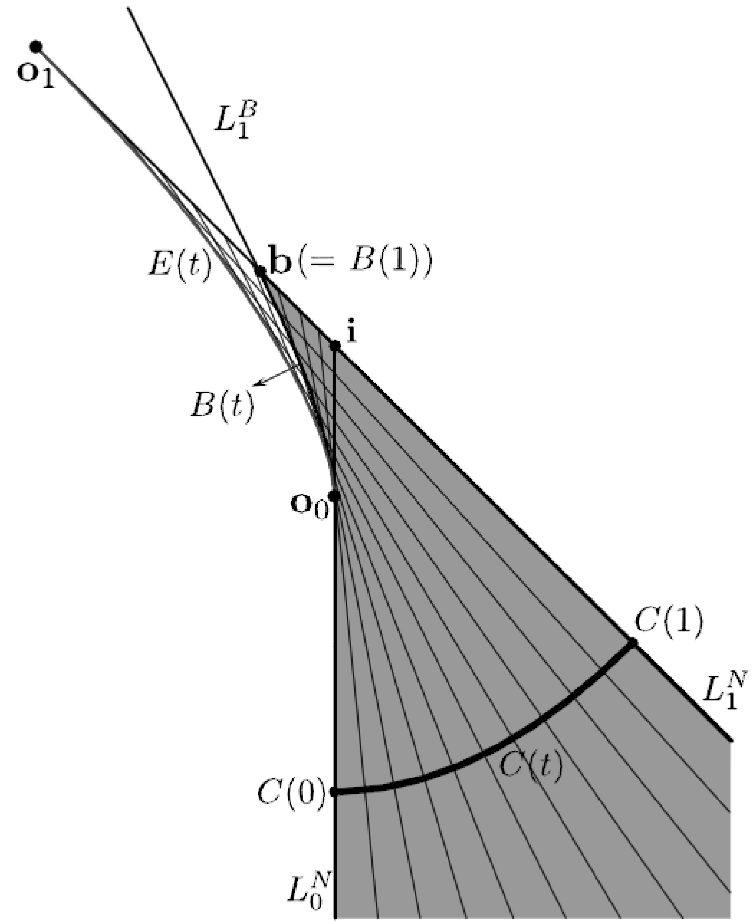
$$\begin{aligned}
x(u,v,s,t) &= x_1(u,v) + x_2(s,t) \\
y(u,v,s,t) &= y_1(u,v) + y_2(s,t) \\
z(u,v,s,t) &= z_1(u,v) + z_2(s,t) \\
F(u,v,s,t) &= \left\langle N_1(u,v), \frac{\partial S_2}{\partial s}(s,t) \right\rangle = 0 \\
G(u,v,s,t) &= \left\langle N_1(u,v), \frac{\partial S_2}{\partial t}(s,t) \right\rangle = 0
\end{aligned}$$

Rational Envelope of Line Sweep

$$a(t)x + b(t)y + c(t) = 0,$$
$$a'(t)x + b'(t)y + c'(t) = 0.$$

$$x = \frac{b(t)c'(t) - b'(t)c(t)}{a(t)b'(t) - a'(t)b(t)},$$

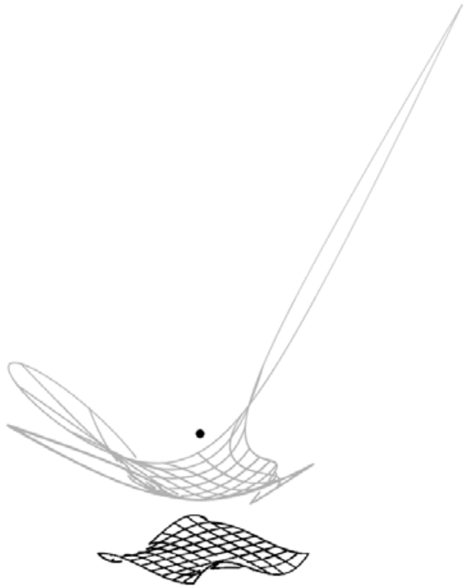
$$y = \frac{a'(t)c(t) - a(t)c'(t)}{a(t)b'(t) - a'(t)b(t)}.$$



Rational Envelope of Plane Sweep

One-parameter family of planes produces a rational developable surface

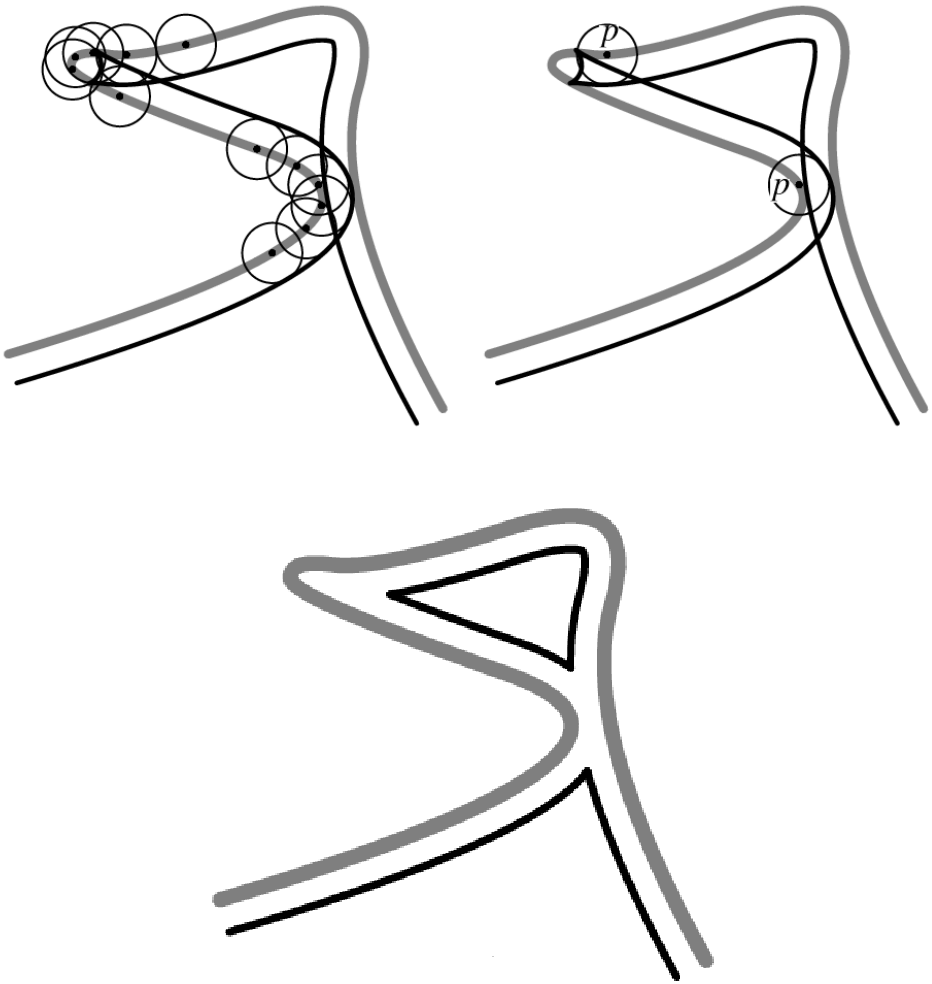
$$a(t)x + b(t)y + c(t)z + d(t) = 0,$$
$$a'(t)x + b'(t)y + c'(t)z + d'(t) = 0.$$

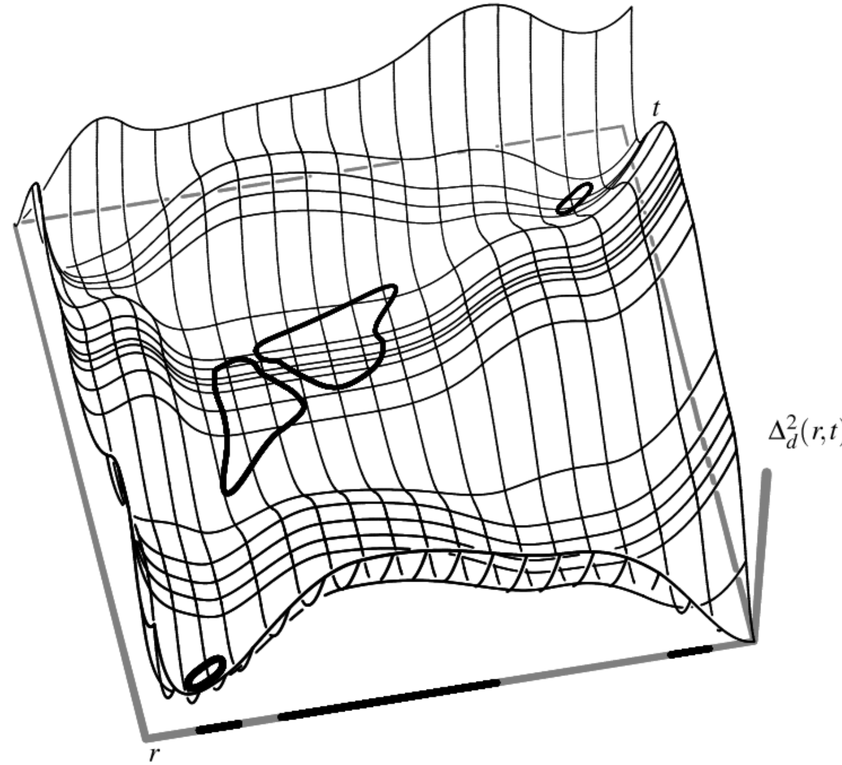


Two-parameter family of planes produces a rational envelope surface

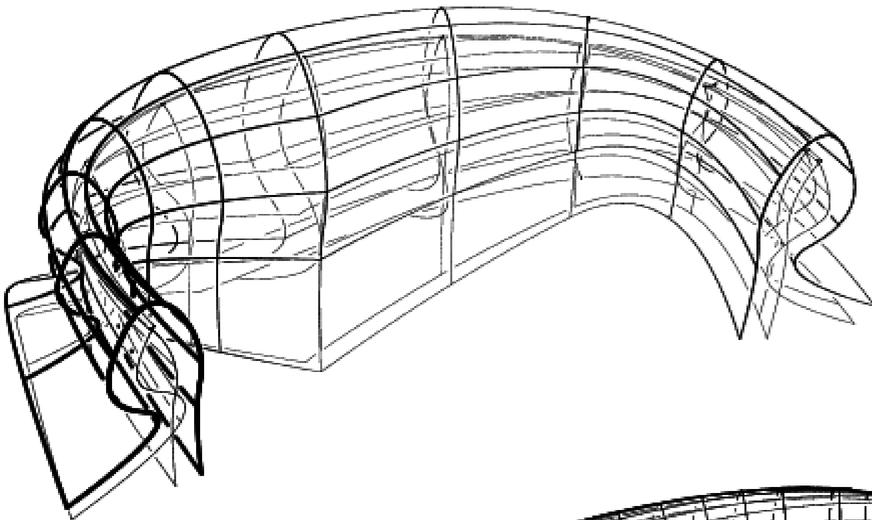
$$a(u,v)x + b(u,v)y + c(u,v)z + d(u,v) = 0,$$
$$a_u(u,v)x + b_u(u,v)y + c_u(u,v)z + d_u(u,v) = 0,$$
$$a_v(u,v)x + b_v(u,v)y + c_v(u,v)z + d_v(u,v) = 0.$$

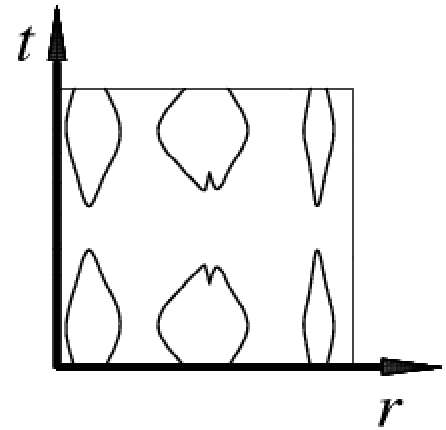
Trimming Offset Curve

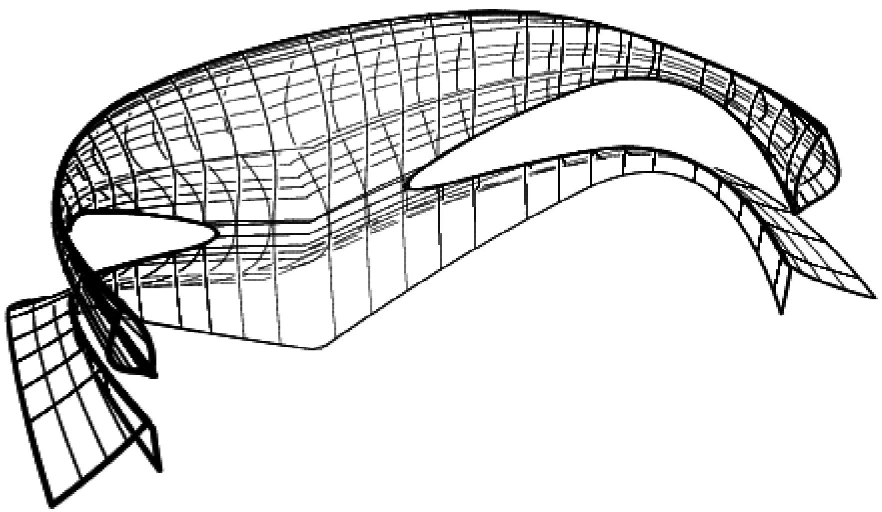




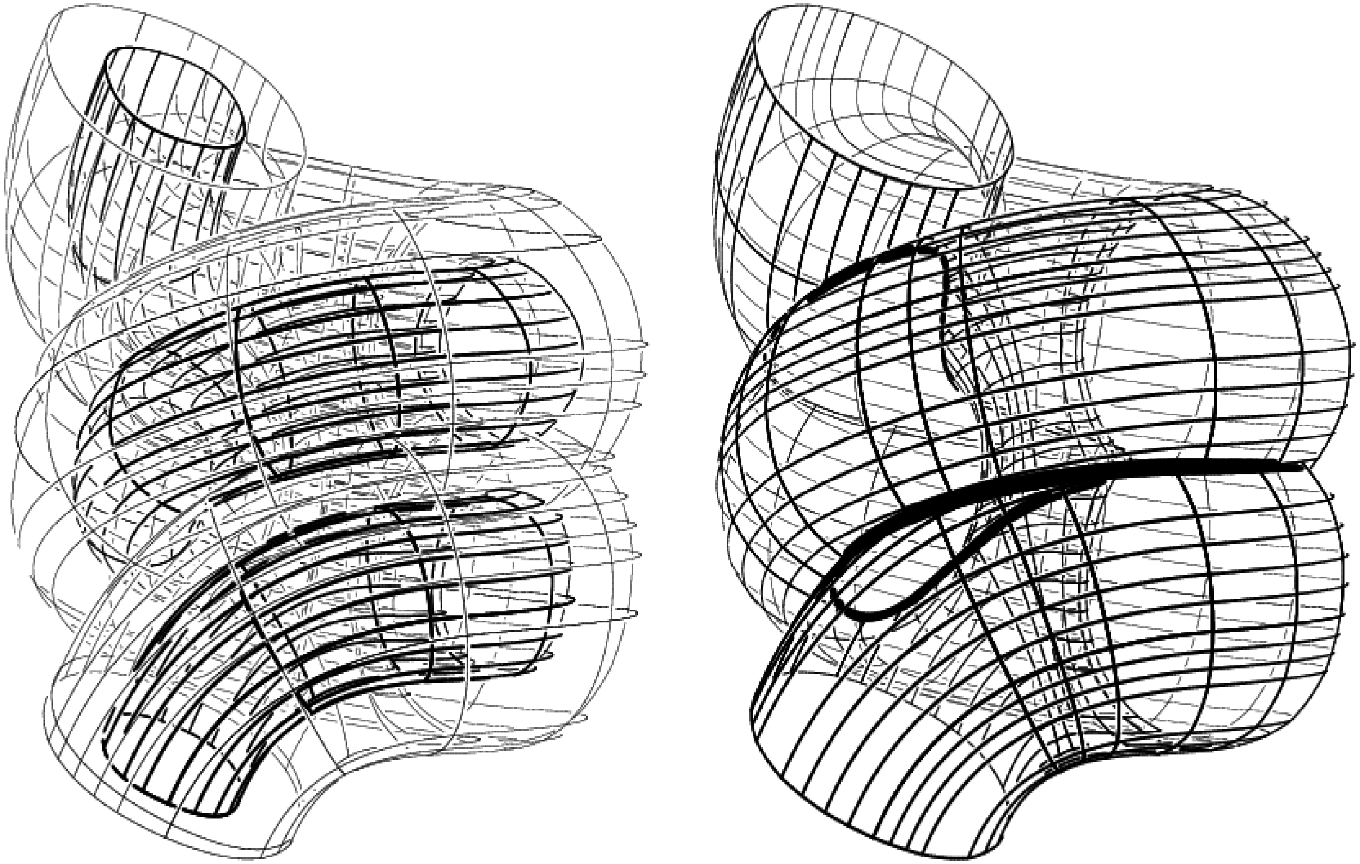
Trimming Offset Surface

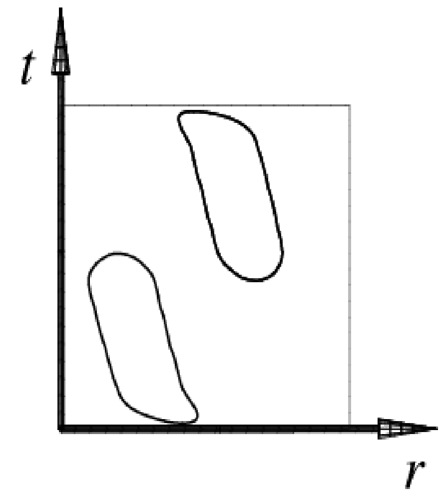






Trimming Offset Surface





Conclusions

- Problem reduction to a system of equations in the parameter space
- Degree reduction in the parameter space
- Dimension reduction to the parameter space
- Squared-distance-based formulation for offset trimming