Engineering Mathematics I

Midterm Exam, October 19, 2016

1. (15 points) A young man with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

   (a) (10 points) Determine the sum $S(t)$ accumulated at any time t.

$$\begin{aligned} S'(t) &= rS(t) + k \\ \frac{dS}{dt} &= rS + k \\ \frac{dS}{S + k/r} &= rdt \\ S + k/r &= \alpha e^{rt} \\ k/r &= \alpha \\ S &= \frac{k}{r}\left(e^{rt} - 1\right) \end{aligned}$$

   (b) (5 points) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

$$\begin{aligned} 1,000,000 &= \frac{k}{0.075}\left(e^{0.075\times 40} - 1\right) \\ k &= \frac{75,000}{e^3 - 1} \end{aligned}$$

2. (20 points) Solve the following initial value problem (without using Laplace transforms):

$$\begin{aligned} y_1' &= y_2 + 2e^t, \quad y_1(0) = 1, \\ y_2' &= -y_1 + 2y_2 + 3e^t, \quad y_2(0) = 1. \end{aligned}$$

Solution:

$\lambda = 1$ is a double root for the characteristic equation $-\lambda(2-\lambda) + 1 = (\lambda - 1)^2 = 0$

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the only eigenvector for the double root $\lambda = 1$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}; \qquad \mathbf{y}_2 = \begin{bmatrix} Ae^t + Bte^t \\ Ce^t + Dte^t \end{bmatrix}, \quad \mathbf{y}_2' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}_2$$

$$\begin{aligned} Ae^t + Be^t + Bte^t &= Ce^t + Dte^t, \\ Ce^t + De^t + Dte^t &= (-A + 2C)e^t + (-B + 2D)te^t \\ A + B = C, \ B = D &\Rightarrow \text{Let } B = D = 1, \quad A = 0, \quad C = 1 \end{aligned}$$

$$\mathbf{y}_2 = \begin{bmatrix} te^t \\ (t+1)e^t \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}, \qquad \mathbf{Y}^{-1} = \begin{bmatrix} (t+1)e^{-t} & -te^{-t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

$$\mathbf{u}'(t) = \mathbf{Y}^{-1}\mathbf{g} = \begin{bmatrix} -t + 2 \\ 1 \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} -\frac{1}{2}t^2 + 2t \\ t \end{bmatrix}$$

$$\begin{aligned} \mathbf{y} &= c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \left(-\frac{1}{2}t^2 + 2t\right)\mathbf{y}_1 + t\mathbf{y}_2 \\ \mathbf{y}(0) &= \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c_1 = 1, \ c_2 = 0 \\ \mathbf{y} &= \left(-\frac{1}{2}t^2 + 2t + 1\right)\begin{bmatrix} e^t \\ e^t \end{bmatrix} + \begin{bmatrix} t^2e^t \\ (t^2 + t)e^t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}t^2 + 2t + 1\right)e^t \\ \left(\frac{1}{2}t^2 + 3t + 1\right)e^t \end{bmatrix} \end{aligned}$$

3. (10 points) Solve the following initial value problem by the power series method. Find the recurrence formula and find the first five nonzero terms in the series.

$$y'' - 2xy' - 2y = 0, \quad y(0) = 1, \ y'(0) = 0.$$

$$\begin{aligned}
y &= \sum_{m=0}^{\infty} a_m x^m \\
y' &= \sum_{m=1}^{\infty} m a_m x^{m-1} \\
-2xy' &= \sum_{m=1}^{\infty} (-2m a_m) x^m \\
y'' &= \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s
\end{aligned}$$

$$\begin{aligned}
a_0 &= y(0) = 1 \\
a_1 &= y'(0) = 0
\end{aligned}$$

$$y'' - 2xy' - 2y = -2a_0 + 2a_2 + \sum_{s=1}^{\infty} \left[(s+2)(s+1)a_{s+2} - 2s a_s - 2a_s\right] x^s$$

$$\begin{aligned}
a_2 = a_0 = 1, \qquad & a_{s+2} = \frac{2}{s+2} a_s, \quad (s = 1, 2, 3, \cdots) \\
a_{2k+2} &= \frac{2}{2k+2} \cdot a_{2k} = \frac{1}{k+1} \cdot a_{2k} = \frac{1}{k+1} \cdot \frac{1}{k} \cdot a_{2(k-1)} = \frac{1}{(k+1)!} \\
a_{2k+1} &= \frac{2}{2k+1} \cdot a_{2k-1} = \frac{2^2}{(2k+1)(2k-1)} \cdot a_{2k-3} = \cdots \\
&= \frac{2^k}{(2k+1)(2k-1)\cdots 3} \cdot a_1 = 0
\end{aligned}$$

$$\begin{aligned}
y &= \sum_{m=0}^{\infty} a_m x^m = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0} \frac{1}{n!}(x^2)^n = e^{x^2} \\
&= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots
\end{aligned}$$

4. (20 points) Consider the following integral equation:

$$y(t) + 2\int_0^t \cos(t-\tau)y(\tau)d\tau = e^{-t}. \tag{1}$$

(a) (5 points) Solve Equation (1) using the Laplace transformation.

$$\begin{aligned}
y(t) + 2\ \cos t\ *\ y(t) &= e^{-t} \\
Y + 2\ \frac{s}{s^2+1}\ Y &= \frac{1}{s+1} \\
\frac{(s+1)^2}{s^2+1}\ Y &= \frac{1}{s+1} \\
Y = \frac{s^2+1}{(s+1)^3} &= \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3} \\
y(t) &= (1-2t+t^2)e^{-t}
\end{aligned}$$

(b) (10 points) By differentiating Equation (1) twice, show that $y(t)$ satisfies the following initial value problem:

$$y'' + 2y' + y = 2e^{-t}, \quad y(0) = 1,\ y'(0) = -3. \tag{2}$$

Solution:

$$y(0) + 2\int_0^0 \cos(0-\tau)y(\tau)d\tau = e^0 \quad \Rightarrow \quad y(0) = 1.$$

$$\begin{aligned}
y(t) + 2\int_0^t (\cos t\cos\tau + \sin t\sin\tau)y(\tau)d\tau &= e^{-t} \\
y(t) + 2\cos t\int_0^t \cos\tau\ y(\tau)d\tau + 2\sin t\int_0^t \sin\tau\ y(\tau)d\tau &= e^{-t} \\
y'(t) - 2\sin t\int_0^t \cos\tau\ y(\tau)d\tau + 2\cos t\cos t\ y(t) + & \\
2\cos t\int_0^t \sin\tau\ y(\tau)d\tau + 2\sin t\sin t\ y(t) &= -e^{-t} \\
y'(t) - 2\int_0^t \sin(t-\tau)y(\tau)d\tau + 2y(t) &= -e^{-t} \\
y'(0) + 2y(0) = -e^0 \ \Rightarrow\ y'(0) = -3 &
\end{aligned}$$

$$\begin{aligned}
y''(t) - 2\int_0^t \cos(t-\tau)y(\tau)d\tau + 2y'(t) &= e^{-t} \\
y''(t) + y(t) - e^{-t} + 2y'(t) = e^{-t} & \\
y''(t) + 2y'(t) + y(t) = 2e^{-t} &
\end{aligned}$$

(c) (5 points) Solve Equation (2) and verify that the solution is the same as in Equation (1).

$$\begin{aligned}
\lambda^2 + 2\lambda + 1 = 0, &\qquad (\lambda+1)^2 = 1 \\
y_h = c_1e^{-t} + c_2te^{-t}, &\qquad y_p = At^2e^{-t} \\
y_p'' + 2y_p + y_p &= (At^2 - 4At + 2A)e^{-t} + 2(-At^2 + 2At)e^{-t} + At^2e^{-t} \\
A &= 1 \\
y = y_h + y_p &= c_1e^{-t} + c_2te^{-t} + t^2e^{-t} \\
y(0) = c_1 = 1, &\qquad y'(0) = -c_1 + c_2 = -3 \quad \Rightarrow c_2 = -2 \\
y &= e^{-t} - 2te^{-t} + t^2e^{-t} = (t-1)^2e^{-t}
\end{aligned}$$

5. (15 points) Using Laplace transforms, solve the following system of differential equations

$$\begin{aligned} y_1' + \ y_2' + y_1 + y_2 &= 1, \qquad y_1(0) = 0, \\ y_1' + 2y_2' + \qquad y_2 &= 0, \qquad y_2(0) = 1. \end{aligned}$$

Solution:

$$\begin{aligned} y_1' + 2y_1 + y_2 &= 2 \\ y_2' - y_1 &= -1 \end{aligned}$$

$$\begin{aligned} sY_1 + 2Y_1 + Y_2 &= \frac{2}{s} \\ sY_2 - 1 - Y_1 &= -\frac{1}{s} \end{aligned}$$

$$\begin{aligned} (s+2)Y_1 + Y_2 &= \frac{2}{s} \\ -Y_1 + sY_2 &= \frac{s-1}{s} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &= \frac{1}{s(s+1)^2} \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ s-1 \end{bmatrix} \\ &= \frac{1}{s(s+1)^2} \begin{bmatrix} s+1 \\ s^2+s \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{1}{s+1} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} y_1(t) &= 1 - e^{-t} \\ y_2(t) &= e^{-t} \end{aligned}$$

6. (10 points) Find the following transformations:

(a) (3 points) $\mathcal{L}^{-1}\left[\frac{d^n}{ds^n}\frac{1}{s^2+\omega^2}\right]$, for $n = 1, 2, 3, \cdots$

Solution:

$$(-t)^n \frac{\sin \omega t}{\omega}$$

(b) (2 points) $\mathcal{L}^{-1}\left[\frac{d^n}{ds^n}\frac{s}{s^2+\omega^2}\right]$, for $n = 1, 2, 3, \cdots$

Solution:

$$(-t)^n \cos \omega t$$

(c) (5 points) $\mathcal{L}\left[te^{at}\sin \omega t\right]$

Solution:

$$-\frac{d}{ds}\left[\frac{\omega}{(s-a)^2+\omega^2}\right] = \frac{2\omega(s-a)}{[(s-a)^2+\omega^2]^2}$$

7. (10 points) Using Laplace transforms, solve the following initial value problem:

$$y'' + y = \delta(t-\pi)e^{2t}, \quad y(0) = 0,\ y'(0) = 1.$$

Solution:

$$\begin{aligned}
s^2Y - 1 + Y &= e^{-\pi(s-2)} \\
(s^2+1)Y &= 1 + e^{2\pi} \cdot e^{-\pi s} \\
Y(s) = \frac{1}{s^2+1} &+ e^{2\pi} \cdot e^{-\pi s} \cdot \frac{1}{s^2+1} \\
y(t) &= \sin t + e^{2\pi} \cdot \sin(t-\pi)u(t-\pi)
\end{aligned}$$