# Lecture Note (4190.410)

## Unit Quaternions

Quaternions were discovered by Sir William Hamilton in 1843 as a generalization of complex numbers. Instead of one imaginary unit $i$, three imaginary units $i, j, k$ are used in quaternions:

$$
\begin{array}{cccccc}
 & 1 \cdot i = i, & 1 \cdot j = j, & 1 \cdot k = k, & i^2 = j^2 = k^2 = -1, & \\
i \cdot j = k, & j \cdot i = -k, & j \cdot k = i, & k \cdot j = -i, & k \cdot i = j, & i \cdot k = -j.
\end{array}
$$

Each quaternion is represented as

$$q = w + xi + yj + zk,$$

where $w, x, y, z$ are real numbers. We may represent the quaternion as a 4-tuple of real numbers: $q = (w, x, y, z)$.

For two quaternions: $q_1 = (w_1, x_1, y_1, z_1)$, $q_2 = (w_2, x_2, y_2, z_2)$, the quaternion addition and multiplication are defined as follows

$$
\begin{aligned}
q_1 + q_2 &= (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2), \\
q_1 \cdot q_2 &= (w_1 w_2 - \langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle, \\
&\qquad w_1(x_2, y_2, z_2) + w_2(x_1, y_1, z_1) + (x_1, y_1, z_1) \times (x_2, y_2, z_2)),
\end{aligned}
$$

where $\langle *, * \rangle$ means the inner product of two three-dimensional vectors.

## Unit Quaternions and 3D Rotations

Unit quaternions are closely related to three-dimensional rotations. Given a unit quaternion $q = (w, x, y, z) \in S^3$, $w^2 + x^2 + y^2 + z^2 = 1$, we can represent it as follows

$$q = (w, x, y, z) = (\cos\theta, \sin\theta(a, b, c)),$$

where

$$
\begin{aligned}
(a, b, c) &= \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \\
\theta &= \arctan\left(\frac{\sqrt{x^2 + y^2 + z^2}}{w}\right).
\end{aligned}
$$

The unit quaternion $q = (\cos\theta, \sin\theta(a, b, c)) \in S^3$ represents the rotation by angle $2\theta$ about $(a, b, c) \in S^2$. For any three-dimensional point $(\alpha, \beta, \gamma) \in R^3$, we can show that

$$(\cos\theta, \sin\theta(a, b, c)) \cdot (0, \alpha, \beta, \gamma) \cdot (\cos\theta, -\sin\theta(a, b, c)) = (0, \bar{\alpha}, \bar{\beta}, \bar{\gamma}),$$

which is the result of rotating $(\alpha, \beta, \gamma)$ by angle $2\theta$ about the axis parallel to $(a, b, c)$.

## The Rotation Matrix

$$
\begin{aligned}
& (w,x,y,z)\cdot(0,\alpha,\beta,\gamma)\cdot(w,-x,-y,-z) \\
= \; & (-(x\alpha+y\beta+z\gamma), w(\alpha,\beta,\gamma)+(x,y,z)\times(\alpha,\beta,\gamma))\cdot(w,-x,-y,-z) \\
= \; & (-w(x\alpha+y\beta+z\gamma)+w(\alpha x+\beta y+\gamma z), \\
& \quad (x\alpha+y\beta+z\gamma)(x,y,z)+w^2(\alpha,\beta,\gamma) \\
& \quad +w(x,y,z)\times(\alpha,\beta,\gamma)-w(\alpha,\beta,\gamma)\times(x,y,z) \\
& \quad -((x,y,z)\times(\alpha,\beta,\gamma))\times(x,y,z)) \\
= \; & (0,(x^2\alpha+xy\beta+xz\gamma, xy\alpha+y^2\beta+yz\gamma, xz\alpha+yz\beta+z^2\gamma) \\
& \quad (w^2\alpha, \qquad w^2\beta, \qquad w^2\gamma) \\
& \quad (2wy\gamma-2wz\beta, \quad 2wz\alpha-2wx\gamma, \quad 2wx\beta-2wy\alpha) \\
& \quad (xy\beta+xz\gamma-z^2\alpha-y^2\alpha, \\
& \qquad\qquad xy\alpha+yz\gamma-x^2\beta-z^2\beta, \\
& \qquad\qquad\qquad xz\alpha+yz\beta-x^2\gamma-y^2\gamma)) \\
= \; & (0,(x^2+w^2-y^2-z^2)\alpha+(2xy-2wz)\beta+(2xz+2wy)\gamma, \\
& \quad (2xy+2wz)\alpha+(y^2+w^2-x^2-z^2)\beta+(2yz-2wx)\gamma, \\
& \quad (2xz-2wy)\alpha+(2yz+2wx)\beta+(w^2+z^2-x^2-y^2)\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix}
&= \begin{bmatrix} x^2+w^2-y^2-z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & y^2+w^2-x^2-z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & w^2+z^2-x^2-y^2 \end{bmatrix}
\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\
&= \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}
\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}
\end{aligned}
$$

For each unit quaternion $q=(w,x,y,z)\in S^3$, let $R_q$ denote the above $3\times 3$ matrix. Then one can check that $R_q$ is a three-dimensional rotational matrix, i.e., $R_q\in SO(3)$:

1. Each row is a unit vector, and each column is a unit vector.
2. Rows are mutually orthogonal each other, and columns are mutually orthogonal each other.
3. The determinant of $R_q$ is 1.

**Remark:**

1. $R_{-q}=R_q$.
2. If $q_1,q_2\in S^3$, then $q_2\cdot q_1\in S^3$.
3. $R_{q_2}R_{q_1}=R_{q_2\cdot q_1}$.

## Quaternion Calculus

Given a unit quaternion curve $q(t) \in S^3$, its derivative $q'(t)$ is given in the following form:

$$q'(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$.

To show this, we define

$$\gamma(h) = q(t+h) \cdot \overline{q(t)},$$

where $\overline{q(t)} = (w(t), -x(t), -y(t), -z(t)) \in S^3$ is the conjugate of $q(t)$. Note that $\gamma(h) \in S^3$ and

$$\gamma(0) = q(t) \cdot \overline{q(t)} = (1, 0, 0, 0).$$

(For each unit quaternion $q$, its conjugate $\overline{q}$ is also the inverse of $q$ with respect to the quaternion multiplication. We can also show that $\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}$.) Thus $\gamma'(0)$ is orthogonal to $(1, 0, 0, 0)$ in $R^4$. Moreover, consider the following relation

$$\begin{aligned} q(t+h) &= q(t+h) \cdot \overline{q(t)} \cdot q(t) \\ &= \gamma(h) \cdot q(t). \end{aligned}$$

Differentiating by $h$, we have

$$q'(t+h) = \gamma'(h) \cdot q(t).$$

Setting $h = 0$ in the above equation, we finally get

$$q'(t) = \gamma'(0) \cdot q(t) = (0, v(t)) \cdot q(t),$$

for some $v(t) \in R^3$. (Note that $\gamma'(0)$ is orthogonal to $(1, 0, 0, 0)$ and it is also dependent on $t$.)

## Angular Velocity

Given a point $p \in R^3$ and a continuous rotation $R_{q(t)}$ implied by a unit quaternion curve $q(t) \in S^3$, the rotated point $p(t) = R_{q(t)}(p)$ is contained in a sphere with radius $\|p\|$ and center $(0, 0, 0)$:

$$(0, p(t)) = q(t) \cdot (0, p) \cdot \overline{q(t)}.$$

Differentiating the above, we get

$$\begin{aligned} (0, p'(t)) &= q'(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{q'(t)} \\ &= (0, v(t)) \cdot q(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{(0, v(t)) \cdot q(t)} \\ &= (0, v(t)) \cdot q(t) \cdot (0, p) \cdot \overline{q(t)} + q(t) \cdot (0, p) \cdot \overline{q(t)} \cdot \overline{(0, v(t))} \\ &= (0, v(t)) \cdot (0, p(t)) + (0, p(t)) \cdot (0, -v(t)) \\ &= (0, v(t) \times p(t) + p(t) \times (-v(t)) \\ &= (0, 2v(t) \times p(t)). \end{aligned}$$

This implies that the angular velocity $\omega(t)$ of the rotation $R_{q(t)}$ is given as follows

$$\omega(t) = 2v(t).$$

## Exercises

**Ex1:** Consider a rotation about axis $(1,1,1)$ by angle $60°$. What is the corresponding $3 \times 3$ rotation matrix?

**Solution:**

$$(a,b,c) = \frac{1}{\sqrt{3}}(1,1,1) \in S^2$$
$$2\theta = 60° \quad \Rightarrow \quad \theta = 30°$$

(The rotation angle $60°$ in the physical world should be divided by 2 in the corresponding quaternion representation!)

$$(\cos\theta, \sin\theta(a,b,c)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \cdot \frac{1}{\sqrt{3}}(1,1,1)\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{3}}(1,1,1)\right)$$

$$w = \frac{\sqrt{3}}{2},\ x = y = z = \frac{1}{2\sqrt{3}}$$
$$2xy = \frac{1}{6},\ 2yz = \frac{1}{6},\ 2xz = \frac{1}{6}$$
$$2wx = \frac{1}{2},\ 2wy = \frac{1}{2},\ 2wz = \frac{1}{2}$$
$$2x^2 = \frac{1}{6},\ 2y^2 = \frac{1}{6},\ 2z^2 = \frac{1}{6}$$
$$R_q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

**Ex2:** Given a rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in SO(3),$$

what is the corresponding unit quaternion $q = (w, x, y, z) \in S^3$ such that $R_q = R$?

**Solution:**

$$\begin{aligned}
r_{11} + r_{22} + r_{33} &= 3 - 4(x^2 + y^2 + z^2) = 3 - 4(1 - w^2) = 4w^2 - 1 \\
w &= \pm\frac{\sqrt{1 + r_{11} + r_{22} + r_{33}}}{2} \\
4wz &= r_{21} - r_{12} \ \Rightarrow\ z = \frac{z_{21} - r_{12}}{4w} \\
4wy &= r_{13} - r_{31} \ \Rightarrow\ y = \frac{z_{13} - r_{31}}{4w} \\
4wx &= r_{32} - r_{23} \ \Rightarrow\ x = \frac{z_{32} - r_{23}}{4w}
\end{aligned}$$