1. (10 points) What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1}, and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$l_{21} = 2 \qquad l_{31} = 3 \qquad l_{32} = 2$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix} = A$$

2. (10 points) Find A^{-1} by elimination on $[A\ I]$:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[A \vdots I] = \left[\begin{array}{ccccc:ccccc} 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc:ccccc} 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc:ccccc} 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc:ccccc} 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc:ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\rightarrow A^{-1}$

3. (25 points)

(a) (8 points) Find the rank of A, and give a basis for its nullspace:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (3 points) The first 3 rows of U are a basis for the row space of A – true or false?

(c) (3 points) Columns 1, 3, 6 of U are a basis for the column space of A – true or false?

(d) (3 points) The four rows of A are a basis for the row space of A – true or false?

(e) (5 points) Find as many linearly independent vectors $\mathbf{b}$ as possible for which $A\mathbf{x} = \mathbf{b}$ has a solution.

(f) (3 points) In elimination on A, what multiple of the third row is subtracted to knock out the fourth row?

(a) rank of $A = 3$ ($\because$ 3 non-zero pivots)

$$U \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\{[-2, 1, 0, 0, 0, 0]^T, [-1, 0, -1, 1, 0, 0]^T,$

$[-5, 0, 0, 0, 1, 0]^T\}$: a basis for $N(A)$.

(b) $C(U^T) = C(A^T)$ and

the first 3 rows of U are a basis for $C(U^T)$

(c) $C(U) \neq C(A)$

(d) The four rows of A are linearly dependent

(e) $\{a_1, a_3, a_6\}$ where $A = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6]$

(f) 2

4. (15 points)

(a) (6 points) Compute the projection matrices $P_i = \mathbf{a}_i\mathbf{a}_i^T/\mathbf{a}_i^T\mathbf{a}_i$, $i = 1,2,3$, onto the lines $\mathbf{a}_1 = (-1,2,2)$, $\mathbf{a}_2 = (2,2,-1)$, and $\mathbf{a}_3 = (2,-1,2)$.

(b) (3 points) Project $\mathbf{b} = (1,0,0)$ to $\mathbf{p}_i$ on the lines through $\mathbf{a}_i$, $i = 1,2,3$.

(c) (6 points) Verify that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{b}$ and $P_1 + P_2 + P_3 = I$.

(a) $$P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}, \quad P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

(b) $\mathbb{P}_1 = \left(\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9}\right)$, $\mathbb{P}_2 = \left(\frac{4}{9}, \frac{4}{9}, -\frac{2}{9}\right)$

$\mathbb{P}_3 = \left(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right)$

(c) $\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 = \left(\frac{9}{9}, 0, 0\right) = \mathbb{b}$

$$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

5. (20 points) We want to fit a plane $z = C + Dx + Ey$ to the four points:

$$z = 3 \quad \text{at} \quad x = 1,\ y = 1; \quad z = 6 \quad \text{at} \quad x = 0,\ y = 3$$
$$z = 5 \quad \text{at} \quad x = 2,\ y = 1; \quad z = 0 \quad \text{at} \quad x = 0,\ y = 0$$

(a) (10 points) Find 4 equations in 3 unknowns to pass a plane through the points (if there is such a plane).

(b) (10 points) Find 3 equations in 3 unknowns for the best least-squares solution.

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 26 \end{bmatrix}$$

6. (10 points) True or false (give an example in either case):

(a) (5 points) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

(b) (5 points) If a 3×2 matrix Q has orthonormal columns, then $\|Q\mathbf{x}\|$ always equals $\|\mathbf{x}\|$.

(a) True

Pf. $Q^T \cdot Q = I \Rightarrow Q^T = Q^T \cdot Q \cdot Q^{-1} = Q^{-1}$

$\therefore Q \cdot Q^T = I \Rightarrow (Q^T)^T \cdot Q^T = I$

$\therefore Q^{-1} = Q^T$ is an orthogonal matrix ⌟

(b) True

Pf. $\|Qx\|^2 = (Qx)^T(Qx) = x^T Q^T Q x$

$= x^T x = \|x\|^2$

$\therefore \|Qx\| = \|x\|$ ⌟

7. (10 points) From the formula $AC^T = (\det A)I$, show that $\det C = (\det A)^{n-1}$.

$$\det(A \cdot C^T) = \det((\det A) I)$$

$$\det(A) \cdot \det(C^T) = (\det A)^n \cdot \det(I)$$

$$\det(A) \cdot \det(C) = (\det A)^n$$

$$\therefore \det(C) = (\det A)^{n-1}$$