Engineering Mathematics I (Comp 400.001)

Solution Set for Midterm Exam I : April 24, 2001

1. Find differential equations that describe the following curves.

   (a) (7 points) A curve in the xy-plane has the property that its slope at each point is equal to the sum of the square of the coordinates of the point.

   $$y' = x^2 + y^2$$

   (b) (8 points) The graph of a nonnegative function has the property that the length of the arc between any two points on the graph is equal to the area of the region under the arc.

   $$\int_a^x \sqrt{1+y'(t)^2}dt = \int_a^x y(t)dt$$

2. (10 points) Find the general solution of the following differential equation:

$$\frac{y^2}{2} + 2ye^x + (y+e^x)\frac{dy}{dx} = 0$$

$$\underbrace{\left(\frac{y^2}{2}+2ye^x\right)}_{P}dx + \underbrace{(y+e^x)}_{Q}dy = 0 \qquad +2$$

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{y+e^x}\left(y+2e^x-e^x\right) = 1 \qquad +2$$

$$F(x) = \exp\left(\int 1\,dx\right) = e^x \qquad +2$$

$$\left(\frac{1}{2}y^2e^x + 2ye^{2x}\right)dx + \left(ye^x + e^{2x}\right)dy = 0 \quad : \text{exact}$$

$$u(x,y) = \frac{1}{2}y^2e^x + ye^{2x} + k(y) \qquad +2$$

$$\frac{\partial u}{\partial y} = ye^x + e^{2x} + k'(y) = ye^x + e^{2x}$$

$$\therefore k(y) = \text{const}$$

$$\therefore u(x,y) = \frac{1}{2}y^2e^x + ye^{2x} + c = 0 \qquad +2$$

3. (10 points) Using a particular solution $y = e^{-x}$, find the general solution of the following differential equation:

$$xy'' + (x-1)y' - y = 0, \ x > 0.$$

$$y'' + \underbrace{\left(1-\frac{1}{x}\right)}_{P(x)} y' - \underbrace{\frac{1}{x}}_{q(x)} y = 0, \ x > 0 \quad +2$$

$$u = \frac{1}{y_1^2} \cdot e^{-\int p(x)dx} = e^{2x} \cdot e^{-(x - \ln x)} = e^{x} \cdot x = xe^{x} \quad +4$$

$$y_2 = y_1 \int u(x)dx = e^{-x} \int xe^{x} dx = e^{-x}\left[xe^{x} - e^{x}\right] = (x-1) \quad +4$$

$$y = k_1 e^{-x} + k_2 (x-1)$$

4. (10 points) Find the general solution of the following equation:

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}.$$

$$\left.\begin{array}{l} \lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3 \\ y_h = c_1 e^{3x} + c_2 x e^{3x} \end{array}\right] +2$$

$$\left.\begin{array}{l} y_p = a + bx + cx^2 + dx^2 e^{3x} \\ y_p' = \quad b + 2cx + 2dx e^{3x} + 3dx^2 e^{3x} \\ y_p'' = \quad 2c + 2de^{3x} + 12dxe^{3x} + 9dx^2 e^{3x} \end{array}\right] +3$$

$$y_p'' - 6y_p' + 9y_p = 9a - 6b + 2c + (9b - 12c)x + 9cx^2 + 2de^{3x}$$

$$\left.\begin{array}{r} 9a - 6b + 2c = 2 \\ 9b - 12c = 0 \\ 9c = 6 \\ 2d = -12 \end{array}\right\} \Rightarrow \begin{array}{l} a = \frac{2}{3} \\ b = \frac{8}{9} \\ c = \frac{2}{3} \\ d = -6 \end{array}$$

$$\therefore y_p = \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2 e^{3x} \quad +5$$

$$\therefore y = y_h + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2 e^{3x}$$

5. (15 points) Find the general solution of the following equation:

$$x^2y'' - 3xy' + 3y = 2x^4e^x.$$

$m(m-1) - 3m + 3 = 0,$

$m = 1, 3$

$y_1 = x, \quad y_2 = x^3$

+3

$y_h = c_1x + c_2x^3$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$ +3

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = \underbrace{2x^2e^x}_{\substack{\parallel \\ r(x)}}$$

$$y_p = -y_1\int\frac{y_2 \cdot r(x)}{W(x)}dx + y_2 \cdot \int\frac{y_1 \cdot r(x)}{W(x)}dx$$ +3

$$= -x\int\frac{x^3 \cdot 2x^2e^x}{2x^3}dx + x^3\int\frac{x \cdot 2x^2e^x}{2x^3}dx$$

$$= -x\int x^2e^x dx + x^3\int e^x dx$$

$$= -x\left(\cancel{x^2e^x} - \int 2xe^x dx\right) + \cancel{x^3e^x}$$

$$= x\left(2xe^x - \int 2e^x dx\right)$$

$$= 2x^2e^x - 2xe^x.$$

+4

$$\therefore y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$$ +2

6. (10 points) Find the Laplace transform of the following function:

$$te^{2t}f'(t)$$

$$\begin{aligned}
\mathcal{L}[f(t)] &= F(s) \\
\mathcal{L}[f'(t)] &= sF(s) - f(0) \\
\mathcal{L}[e^{2t}f'(t)] &= (s-2)F(s-2) - f(0) \\
\mathcal{L}[t(e^{2t}f'(t))] &= -\frac{d}{ds}[(s-2)F(s-2) - f(0)] \\
&= -F(s-2) + (2-s)F'(s-2)
\end{aligned}$$

+3

+3

+4

7. (15 points) Find the Laplace transform of the following function:

$$t^2\left(\int_0^t \tau \sin\tau d\tau\right)$$

$$\begin{aligned}
\mathcal{L}[\sin t] &= \frac{1}{s^2+1} \\
\mathcal{L}[t\sin t] &= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2} \\
\mathcal{L}[\int_0^t \tau\sin\tau d\tau] &= \frac{1}{s}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2} \\
\mathcal{L}[t^2\left(\int_0^t \tau\sin\tau d\tau\right)] &= \frac{d^2}{ds^2}\left[\frac{2}{(s^2+1)^2}\right] \\
&= \frac{d}{ds}\left[\frac{-4(2s)}{(s^2+1)^3}\right] = \frac{-8(s^2+1)^3 + 8s\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6} \\
&= \frac{-8s^2 - 8 + 48s^2}{(s^2+1)^4} = \frac{40s^2-8}{(s^2+1)^4}
\end{aligned}$$

+2

+2

+4

+4

+3

8. (15 points) Find f and g satisfying the following simultaneous equations:

$$f(t) + \int_0^t (t-\tau)g(\tau)d\tau = \sin 2t$$
$$g(t) + \int_0^t (t-\tau)f(\tau)d\tau = 0$$

+5

$$\left\{\begin{aligned}
F(s) + \tfrac{1}{s^2}G(s) &= \tfrac{2}{s^2+4} \\
\tfrac{1}{s^2}F(s) + G(s) &= 0
\end{aligned}\right. \Longrightarrow \left\{\begin{aligned}
s^2F(s) + G(s) &= \tfrac{2s^2}{s^2+4} \\
F(s) + s^2G(s) &= 0
\end{aligned}\right.$$

$$\begin{bmatrix} F(s) \\ G(s) \end{bmatrix} = \frac{1}{s^4-1}\begin{bmatrix} s^2 & -1 \\ -1 & s^2 \end{bmatrix}\begin{bmatrix} \frac{2s^2}{s^2+4} \\ 0 \end{bmatrix}$$

+3

$$\begin{aligned}
F(s) &= \frac{2s^4}{(s^4-1)(s^2+4)} = \frac{1}{10}\cdot\frac{1}{s-1} - \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{16}{15}\cdot\frac{2}{s^2+4} \\
G(s) &= \frac{-2s^2}{(s^4-1)(s^2+4)} = -\frac{1}{10}\cdot\frac{1}{s-1} + \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{4}{15}\cdot\frac{2}{s^2+4}
\end{aligned}$$

+4

$$\begin{aligned}
f(t) &= \frac{1}{5}\sinh t - \frac{1}{3}\sin t + \frac{16}{15}\sin 2t \\
g(t) &= -\frac{1}{5}\sinh t - \frac{1}{3}\sin t + \frac{4}{15}\sin 2t
\end{aligned}$$

+3