Engineering Mathematics I
(Comp 400.001)

Midterm Exam I: October 11, 2004

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Name: ____________________

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1. (20 points) A compound C is formed when two chemicals A and B are combined. The reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. Determine the amount of C, denoted by $x(t)$, at any time t if the rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially there are 50 grams of A and 32 grams of B.

   (a) (4 points) Let $a(t)$ denote the number of grams of the compound A present at time t, and $b(t)$ denote that of the compound B. Represent $a(t)$ and $b(t)$ using $x(t)$, the amount of C at time t.

   (b) (6 points) Set up a differential equation for $x(t)$.

   (c) (8 points) It is observed that 30 grams of the compound C is formed in 10 minutes. Solve the differential equation to determine $x(t)$ as a function of t.

   (d) (2 points) What is the limiting behavior of the solution $x(t)$ as $t \to \infty$?

(a) $a(t) = 50 - 0.2\,x(t)$, $b(t) = 32 - 0.8\,x(t)$ +2 +2

(b) $x'(t) = k_1 a(t) b(t) = k_1 (50 - 0.2x(t))(32 - 0.8x(t))$

$= k_2 (x(t) - 250)(x(t) - 40)$, where $k_2 = \frac{8}{50} k_1$ +6

(c)

$$\frac{dx}{(x-250)(x-40)} = k_2 dt \quad +2$$

$$\frac{dx}{x-250} - \frac{dx}{x-40} = k_3 dt, \quad \text{where } k_3 = 210 k_2$$

$$\ln \left| \frac{x-250}{x-40} \right| = k_3 t + \alpha$$

$$\frac{x-250}{x-40} = c_1 \cdot e^{k_3 t}, \quad \text{where } c_1 = e^{\alpha} \quad +2$$

Since $x(0) = 0$ and $x(10) = 30$

$c_1 = \frac{25}{4}$ +1, and $22 = \frac{25}{4} e^{k_3 \times 10}$

$\therefore k_3 = \frac{1}{10} \ln \frac{88}{25}$ +1

$$x(t) = \frac{40 c_1 e^{k_3 t} - 250}{c_1 e^{k_3 t} - 1}$$

$$= \frac{1000 (e^{k_3 t} - 1)}{25 e^{k_3 t} - 4}, \quad \text{where} \quad +2$$

(d) $\lim_{t \to \infty} x(t) = \frac{1000}{25} = 40$ +2

2. (10 points) The differential equation

$$\left(x - \sqrt{x^2 + y^2}\right) dx + y\ dy = 0$$

is not exact, but show how the rearrangement

$$x\ dx + y\ dy = \sqrt{x^2 + y^2}\ dx$$

and the observation

$$\frac{1}{2}\ d(x^2 + y^2) = x\ dx + y\ dy$$

leads to a solution.

$$\frac{1}{2} d(x^2+y^2) = \sqrt{x^2+y^2}\, dx \quad +1$$

Let $t = x^2+y^2$, then $\quad +3$

$$\frac{1}{2} dt = \sqrt{t}\, dx \quad +1$$

$$(t)^{-\frac{1}{2}} dt = 2dx \quad +2$$

$$2\sqrt{t} = 2x + C_1 \quad +2$$

$$\therefore \sqrt{x^2+y^2} - x = C \quad \text{for some } C = \frac{C_1}{2}.$$

+1

3. (15 points) Solve the following initial value problem

$$y'' + y = \cos t, \quad y(0) = 1, \; y'(0) = 1.$$

$\lambda^2 + 1 = 0, \quad \lambda = \pm i$

$y_G = c_1 \cos t + c_2 \sin t$ (+2)

$y_p = At\cos t + Bt\sin t$ (+5)

$y_p' = A\cos t + B\sin t - At\sin t + Bt\cos t$

$y_p'' = -2A\sin t + 2B\cos t - At\cos t - Bt\sin t$

$y_p'' + y_p = -2A\sin t + 2B\cos t = \cos t$

$\therefore A = 0, \quad B = \frac{1}{2}$

(+2)

$y = y_G + y_p = c_1\cos t + c_2 \sin t + \frac{1}{2}t\sin t$ (+2)

$y' = -c_1\sin t + c_2\cos t + \frac{1}{2}\sin t + \frac{1}{2}t\cos t$ (+1)

$y(0) = c_1 = 1, \quad y'(0) = c_2 = 1$ (+2)

$\therefore y = \cos t + \sin t + \frac{1}{2}t\sin t$ (+1)

4. (10 points) Show that

$$\mathcal{L}^{-1}\left[F^{(n)}(s)\right] = (-t)^n f(t), \quad \text{for } n = 1, 2, 3, \cdots$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \int_0^\infty e^{-st} (-t) f(t)\,dt$$

+3

Assume $F^{(k)}(s) = \int_0^\infty e^{-st} (-t)^k f(t)\,dt$

Then $F^{(k+1)}(s) = \int_0^\infty e^{-st} (-t)^{k+1} f(t)\,dt$

+6

$\therefore F^{(n)}(s) = \int_0^\infty e^{-st} (-t)^n f(t)\,dt$, for $n = 1, 2, \cdots$

$\mathcal{L}^{-1}\left[F^{(n)}(s)\right] = (-t)^n f(t)$, for $n = 1, 2, \cdots$

+1

5. (10 points) Find the Laplace transform of the following function:

$$t\Big(\int_0^t \tau^2 \cos\tau\, d\tau\Big)$$

$$\mathcal{L}[\cos t] = \frac{s}{s^2+1} \quad +1$$

$$\mathcal{L}[t^2\cos t] = \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right] = \frac{d}{ds}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right] \quad +2$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \frac{-2s(s^2+1)^2 - (1-s^2)\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3-6s}{(s^2+1)^3} \quad +2$$

$$\mathcal{L}\left[\int_0^t \tau^2\cos\tau\, d\tau\right] = \frac{2s^2-6}{(s^2+1)^3} \quad +2$$

$$\mathcal{L}\left[t\int_0^t \tau^2\cos\tau\, d\tau\right] = -\frac{d}{ds}\left[\frac{2s^2-6}{(s^2+1)^3}\right]$$

$$= -\frac{4s(s^2+1)^3 - (2s^2-6)\,3(s^2+1)^2\cdot 2s}{(s^2+1)^6}$$

$$= -\frac{4s^3+4s-12s^3+36s}{(s^2+1)^4}$$

$$= \frac{8s^3-40s}{(s^2+1)^4} \quad +3$$

6. (20 points) Solve the following initial value problem

$$y'' + y = \begin{cases} 1, & \text{if } 0 < t < \frac{\pi}{2}, \\ \sin t, & \text{if } \frac{\pi}{2} < t < \infty; \end{cases} \qquad y(0) = 1,\ y'(0) = 0$$

$$y'' + y = 1 + (\sin t - 1)\,u(t - \tfrac{\pi}{2}) \quad +3$$

$$= 1 + \left[\cos(t - \tfrac{\pi}{2}) - 1\right] u(t - \tfrac{\pi}{2}) \quad +4$$

$$s^2 Y - s + Y = \frac{1}{s} + \left(\frac{s}{s^2+1} - \frac{1}{s}\right) e^{-\frac{\pi}{2}s}$$

$$(s^2+1)Y = s + \frac{1}{s} + \left[\frac{s}{s^2+1} - \frac{1}{s}\right] e^{-\frac{\pi}{2}s} \quad +3$$

$$Y = \frac{s}{s^2+1} + \frac{1}{s(s^2+1)} + \left[\frac{s}{(s^2+1)^2} - \frac{1}{s(s^2+1)}\right] e^{-\frac{\pi}{2}s}$$

$$= \cancel{\frac{s}{s^2+1}} + \frac{1}{s} - \cancel{\frac{s}{s^2+1}}$$

$$+ \left[\frac{s}{(s^2+1)^2} - \frac{1}{s} + \frac{s}{s^2+1}\right] e^{-\frac{\pi}{2}s} \quad +3$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \int_0^t \sin\tau \cdot \cos(t-\tau)\, d\tau$$

$$= \frac{1}{2} t \sin t \quad +4$$

$$y(t) = 1 + \left(\frac{1}{2}(t - \tfrac{\pi}{2})\sin(t - \tfrac{\pi}{2}) - 1 + \cos(t - \tfrac{\pi}{2})\right) u(t - \tfrac{\pi}{2})$$

$$= 1 + \left[\frac{1}{2}(t - \tfrac{\pi}{2})(-\cos t) - 1 + \sin t\right] u(t - \tfrac{\pi}{2})$$

+3

7. (15 points) Use the Laplace transform to solve the following system of differential equations

$$\begin{aligned} y_1''(t) + 3y_2'(t) + 3y_2(t) &= 0, \quad y_1(0) = 0,\ y_1'(0) = 2 \\ y_1''(t) \qquad\qquad + 3y_2(t) &= te^{-t}, \quad y_2(0) = 0. \end{aligned}$$

$$3y_2'(t) = -te^{-t} \quad +2$$

$$3sY_2 = -\frac{1}{(s+1)^2} \quad +2$$

$$Y_2 = \frac{-1}{3s(s+1)^2} = \frac{-1/3}{s} + \frac{1/3}{s+1} + \frac{1/3}{(s+1)^2} \quad +2$$

$$y_2(t) = -\frac{1}{3} + \frac{1}{3}e^{-t} + \frac{1}{3}te^{-t} \quad +2$$

$$y_1''(t) = te^{-t} - 3y_2(t) = 1 - e^{-t} \quad +1$$

$$s^2Y_1 - 2 = \frac{1}{s} - \frac{1}{(s+1)} \quad +2$$

$$\begin{aligned} Y_1 &= \frac{2}{s^2} + \frac{1}{s^3} - \frac{1}{s^2(s+1)} \\ &= \frac{2}{s^2} + \frac{1}{s^3} - \left(\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right) \\ &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1} \quad +2 \end{aligned}$$

$$y_1(t) = 1 + t + \frac{1}{2}t^2 - e^{-t} \quad +2$$