Quiz #4 (CSE 400.001)

Monday, October 25, 2004

1. (7 points) Use $x_0 = 2.100$ and $x_1 = 2.009$ in solving the following equation by Newton's method
$$x^5 - 32 = 0.$$
How many additional iterations are necessary to produce the solution to 20D accuracy?

Solution:
$$\frac{f''(s)}{2f'(s)} \approx \frac{f''(x_1)}{2f'(x_1)} = \frac{20x_1^3}{10x_1^4} = \frac{2}{x_1} \approx 0.9955$$
$$|\epsilon_{n+1}| \approx 0.9955\epsilon_n^2 \approx 0.9955^3\epsilon_{n-1}^4 \approx 0.9955^{2^{n+1}-1}\epsilon_0^{2^{n+1}} \leq 5 \cdot 10^{-21}$$
$$\epsilon_1 - \epsilon_0 = (\epsilon_1 - s) - (\epsilon_0 - s) = -x_1 + x_0 \approx 0.091$$
$$\epsilon_1 \approx \epsilon_0 + 0.091 \approx -0.9955\epsilon_0^2$$
$$0.9955\epsilon_0^2 + \epsilon_0 + 0.091 \approx 0$$
$$\epsilon_0 \approx -0.1012$$

$$\begin{aligned}
n = 1: &\quad 0.9955^3 \cdot 0.1012^4 \approx 1.034 \cdot 10^{-4} > 5 \cdot 10^{-21} \\
n = 2: &\quad 0.9955^7 \cdot 0.1012^8 \approx 1.066 \cdot 10^{-8} > 5 \cdot 10^{-21} \\
n = 3: &\quad 0.9955^{15} \cdot 0.1012^{16} \approx 1.132 \cdot 10^{-16} > 5 \cdot 10^{-21} \\
n = 4: &\quad 0.9955^{31} \cdot 0.1012^{32} < 10^{-30} < 5 \cdot 10^{-21}
\end{aligned}$$

Hence, $n = 4$ additional iterations are necessary.

2. (4 points) Interpolate
$$f_0 = f(0) = 0,\ f_1 = f(1) = 12,\ f_2 = f(2) = 6,\ f_3 = f(3) = 0$$
by the cubic spline satisfying $k_0 = 3$ and $k_3 = -6$.

Solution:
$$\left\{\begin{array}{rcl} k_0 + 4k_1 + k_2 &=& 3 \cdot (6) = 18 \\ k_1 + 4k_2 + k_3 &=& 3 \cdot (-12) = -36 \end{array}\right. \implies \left\{\begin{array}{rcl} 4k_1 + k_2 &=& 15 \\ k_1 + 4k_2 &=& -30 \end{array}\right. \implies k_1 = 6, \quad k_2 = -9$$
$$\left\{\begin{array}{rcl} p_0(x) &=& -15x^3 + 24x^2 + 3x, \quad \text{for } 0 \leq x \leq 1 \\ p_1(x) &=& 9(x-1)^3 - 21(x-1)^2 + 6(x-1) + 12, \quad \text{for } 1 \leq x \leq 2 \\ p_2(x) &=& -3(x-2)^3 + 6(x-2)^2 - 9(x-2) + 6, \quad \text{for } 2 \leq x \leq 3 \end{array}\right.$$

3. (4 points) Compute the following integral using the Gauss quadrature with $n = 5$.
$$\int_0^{\frac{\pi}{4}} \tan x^3 \, dx$$

Solution:
$$x = \frac{\pi}{8}(t+1) \quad \Rightarrow \quad dx = \frac{\pi}{8}dt$$
$$\frac{\pi}{8}\int_{-1}^{1} \tan \frac{\pi^3(t+1)^3}{8^3}dt = \cdots$$