Engineering Mathematics I
(Comp 400.001)

Midterm Exam II: May 18, 2004

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Name: ______________________

ID No: ______________________

Dept: ______________________

E-mail: ______________________

1. (15 points) Find the cubic spline for the data

$$f(-1) = 3,\ f(1) = 1,\ f(3) = 23,\ f(5) = 45,\ k_0 = k_3 = 3.$$

$$\left.\begin{aligned} k_0 + 4k_1 + k_2 &= \frac{3}{2} \times 20 = 30 \\ k_1 + 4k_2 + k_3 &= \frac{3}{2} \times 44 = 66 \end{aligned}\right] \Rightarrow \begin{cases} 4k_1 + k_2 = 27 \\ k_1 + 4k_2 = 63 \end{cases}$$

+4

$\therefore k_1 = 3,\ k_2 = 15$

+2

$$\begin{cases} P_0(x) = 2(x+1)^3 - 6(x+1)^2 + 3(x+1) + 3, & \text{for } -1 \leq x \leq 1 \\ P_1(x) = -(x-1)^3 + 6(x-1)^2 + 3(x-1) + 1, & \text{for } 1 \leq x \leq 3 \\ P_2(x) = -(x-3)^3 + 15(x-3) + 23, & \text{for } 3 \leq x \leq 5 \end{cases}$$

2. (10 points) Fit a straight line by least squares to

$$(-2,-6),\ (-1,-2),\ (0,-1),\ (1,0),\ (2,10),\ (4,78).$$

$$\left.\begin{array}{l} n=6, \quad \sum x_j = 4, \quad \sum x_j^2 = 26 \\ \qquad\quad \sum y_j = 79, \quad \sum x_j y_j = 346 \end{array}\right] \quad +5$$

$$\begin{bmatrix} 6 & 4 \\ 4 & 26 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 79 \\ 346 \end{bmatrix} \quad +2$$

$$a = \frac{67}{14}, \quad b = \frac{176}{14} \quad +2$$

$$y = \frac{67}{14} + \frac{176}{14} x \quad +1$$

3. (15 points) Solve the following linear system by Cholesky's method. Show the details of your work, in particular, the LU-factorization.

$$\begin{aligned} 9x_1 + 12x_2 + \ 6x_3 &= 57 \\ 12x_1 + 17x_2 + 11x_3 &= 81 \\ 6x_1 + 11x_2 + 17x_3 &= 57 \end{aligned}$$

$$\begin{bmatrix} 9 & 12 & 6 \\ 12 & 17 & 11 \\ 6 & 11 & 17 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$l_{11} = \sqrt{9} = 3$ (+1)

$l_{21} = 12/3 = 4, \quad l_{31} = 6/3 = 2$ (+2)

$l_{22} = \sqrt{17 - 4^2} = \sqrt{1} = 1$ (+2)

$l_{32} = \frac{1}{l_{22}}(11 - l_{31} \cdot l_{21}) = 11 - 8 = 3$ (+2)

$l_{33} = \sqrt{17 - (l_{31}^2 + l_{32}^2)} = \sqrt{4} = 2$ (+2)

$$\begin{bmatrix} 9 & 12 & 6 \\ 12 & 17 & 11 \\ 6 & 11 & 17 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow$ Solving $Ly = \mathbb{b}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 57 \\ 81 \\ 57 \end{bmatrix} \Rightarrow y_1 = 19, \ y_2 = 5, \ y_3 = 2$$

(+3)

Solving $L^T x = y$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \\ 2 \end{bmatrix} \Rightarrow x_3 = 1, \ x_2 = 2, \ x_1 = 3$$

(+3)

4. (10 points) Table 1 shows the result of applying the Improved Euler method to the following initial value problem with $h = 0.2$:

$$y' = -(y+1)(y+3), \quad \text{for } 0 \le x \le 1, \quad y(0) = -2.$$

Fill in the blank; and show your work for partial credit.

x_i	y_i
0.2	−1.80400
0.4	−1.62292
0.6	−1.46724
0.8	−1.34132
1.0	−1.24429

Table 1: Improved Euler Method

$x_1 = 0.2, \quad y_1 = -1.80400$

$h = 0.2, \quad f(x,y) = -(y+1)(y+3)$ +2

$k_1 = 0.2 * f(x_1, y_1) = 0.192317$ +2

$k_2 = 0.2 * f(x_1 + 0.2, y_1 + k_1) = 0.169842$ +3

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = -1.62292$ +2

5. (20 points) Show that in the case of Figure 428

$$\nabla^2 u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq}u_O\right]$$

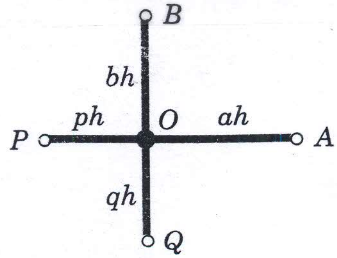

Fig. 428. Neighboring points A, B, P, Q of a mesh point O and notations in formula (6)

$$u_A = u_o + ah\frac{\partial u_o}{\partial x} + \frac{1}{2}(ah)^2\frac{\partial^2 u_o}{\partial x^2} + \cdots$$

$$u_p = u_o - ph\frac{\partial u_o}{\partial x} + \frac{1}{2}(ph)^2\frac{\partial^2 u_o}{\partial x^2} + \cdots$$

+2

$$p u_A + a u_p \approx (p+a)u_o + \frac{1}{2}ap(a+p)h^2\frac{\partial^2 u_o}{\partial x^2}$$

+3

$$\frac{\partial^2 u_o}{\partial x^2} \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_p}{p(p+a)} - \frac{u_o}{ap}\right]$$

+3

Similarly,

$$\frac{\partial^2 u_o}{\partial y^2} \approx \frac{2}{h^2}\left[\frac{u_B}{b(b+q)} + \frac{u_Q}{q(q+b)} - \frac{u_o}{bq}\right]$$

+8

$$\nabla^2 u_o \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_p}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq}u_o\right]$$

+4

6. (30 points) Consider the following hyperbolic equation

$$u_{tt} = u_{xx}, \quad 0 \le x \le 1,\ 0 \le t \le 0.4,$$

with intial and boundary conditions

$$u(x,0) = x^2, \quad u_t(x,0) = 2x; \quad u_x(0,t) = 2t, \quad u(1,t) = (1+t)^2,$$

Approximate the solution to above equation with $h = k = 0.2$, for $0 \le t \le 0.4$.

(a) (5 points) Represent $u_{i,j+1}$ in terms of $u_{i-1,j}, u_{i,j}, u_{i+1,j}, u_{i,j-1}$.

(b) (5 points) Represent $u_{i,1}$ in terms of $u_{i-1,0}, u_{i,0}, u_{i+1,0}$.

(c) (5 points) Represent $u_{0,j+1}$ in terms of $u_{0,j}, u_{1,j}, u_{0,j-1}$.

(d) (5 points) Represent $u_{0,1}$ in terms of $u_{0,0}, u_{1,0}$.

(e) (10 points) Find the values of $u_{i,j}$, for $i = 0,1,2,3,4$, and $j = 1,2$.

(a) $\frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$ (+3)

$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$ (+2)

(b) $u_{i,1} - u_{i,-1} = 2k\cdot(2hi) = 0.16i$ (+2)

$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,1} + 0.16i$ (+2)

$u_{i,1} = \frac{1}{2}(u_{i+1,0} + u_{i-1,0}) + 0.08i$ (+1)

(c) $u_{1,j} - u_{-1,j} = 2h\cdot(2kj) = 0.16j$ (+2)

$u_{0,j+1} + u_{0,j-1} = u_{1,j} + u_{-1,j} = 2u_{1,j} - 0.16j$ (+2)

$u_{0,j+1} = 2u_{1,j} - u_{0,j-1} - 0.16j$ (+1)

(d) $u_{0,1} - u_{0,-1} = 0$ (+2)

$u_{0,1} = 2u_{1,0} - u_{0,-1} = 2u_{1,0} - u_{0,1}$ (+2)

$\therefore u_{0,1} = u_{1,0}$ (+1)

(e)

$u_{i,2} \Rightarrow$	0.16	0.36	0.64	1.00	1.44	1.96
$u_{i,1} \Rightarrow$	0.04	0.16	0.36	0.64	1.00	1.44
$u_{i,0} \Rightarrow$	0	0.04	0.16	0.36	0.64	1.0