Quiz #3 (CSE 400.001)

Wednesday, October 15, 2014

1. (10 points) Solve the following equation using the Power Series Method:

$$y'' + x^2y' + xy = 0.$$

(+1)

$$y = \sum_{m=0}^{\infty} a_m x^m, \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0 \quad (+1)$$

$$\sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s + \sum_{s=2}^{\infty} (s-1) a_{s-1} x^s + \sum_{s=1}^{\infty} a_{s-1} x^s = 0 \quad (+1)$$

$$2a_2 + (6a_3 + a_0)x + \sum_{s=2}^{\infty} \left[(s+2)(s+1) a_{s+2} + s\, a_{s-1} \right] x^s = 0$$

(+1)

$$\therefore a_2 = 0, \quad a_3 = -\frac{1}{6} a_0 \quad (+1)$$

$$a_{s+2} = -\frac{s}{(s+2)(s+1)} a_{s-1} = -\frac{s^2}{(s+2)(s+1)s} a_{s-1} \quad (+1)$$

for $s = 2, 3, 4, \cdots$

$$\begin{cases} a_{3k} = (-1)^k \dfrac{(3k-2)^2 \cdots 1^2}{(3k)!} a_0 \\ a_{3k+1} = (-1)^k \dfrac{(3k-1)^2 \cdots 2^2}{(3k+1)!} a_1 \\ a_{3k+2} = 0 \end{cases}$$

for $k = 1, 2, 3, \cdots$

(+2)

$$y = a_0 \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{(3k-2)^2 \cdots 1^2}{(3k)!} x^{3k} \right]$$

$$+ a_1 \left[x + \sum_{k=1}^{\infty} (-1)^k \frac{(3k-1)^2 \cdots 2^2}{(3k+1)!} x^{3k+1} \right]$$

(+2)

2. (10 points) Solve the following initial value problem:

$$\begin{aligned} y_1' &= y_1 + 8y_2 + 12t, \quad y_1(0) = 2, \\ y_2' &= y_1 - y_2 + 12t, \quad y_2(0) = 0. \end{aligned}$$

$A = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix}$, $\det(A - \lambda I) = (\lambda - 3)(\lambda + 3) = 0$

$\therefore \lambda_1 = 3, \ \lambda_2 = -3$ +1

$\lambda_1 = 3, \ x^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$; $\lambda_2 = -3, \ x^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ +1

$y = y_h + y_p = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} at+b \\ ct+d \end{bmatrix}$ +2

$y_p' = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} at+b \\ ct+d \end{bmatrix} + \begin{bmatrix} 12t \\ 12t \end{bmatrix}$

$\therefore y_p = \begin{bmatrix} -12t - 4/3 \\ -4/3 \end{bmatrix}$ +2

$y(0) = \begin{bmatrix} 4c_1 + 2c_2 - 4/3 \\ c_1 - c_2 - 4/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

+2

$\therefore c_1 = 1, \ c_2 = -\frac{1}{3}$

$y = \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{3} \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -12t - 4/3 \\ -4/3 \end{bmatrix}$

+2